

## Extension of CR-functions on wedges

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**Abstract.** We generalize the extension theorem of Ajrapetyan-Henkin [AH81] and Boggess-Polking [BP82] to wedge-like domains  $V$  in a generic manifold  $M$ . For each complex tangent vector to  $M$ , we introduce its complex angle with respect to  $V$  and show that CR-functions on  $V$  have natural extension in the direction of the Levi form of any tangent vector whose angle is greater than  $\pi/2$ . This gives an explanation of the extension phenomenon recently discovered by Eastwood-Graham [EG99, EG01]. We further demonstrate by an example that the angle  $\pi/2$  is optimal and cannot be replaced by any smaller angle. We conclude by giving applications to holomorphic extension to a full neighborhood of the reference point.

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### 1. Introduction

The celebrated theorem of Ajrapetyan-Henkin [AH81] and Boggess-Polking [BP82] extending classical results of Lewy [L56] states that all (continuous) CR-functions on a generic submanifold  $M$  in  $\mathbb{C}^N$  extend holomorphically to a wedge in the direction of the convex cone spanned by the values of the (vector-valued) Levi form of  $M$ . Here one starts with a submanifold  $M$  and ends with a wedge. A natural question is to obtain a result generalizing the above one within the category of wedges with description of the additional directions of extendibility (see Tumanov [T95a, T95b] for general extension results to wedges without prescription of the direction).

Given a wedge  $V$  in a submanifold  $M$  of  $\mathbb{C}^N$  (see §3 for the definition) whose edge  $E \subset \partial V$  at a point  $p \in E$  is a generic (i.e.  $T_p E + iT_p E = T_p \mathbb{C}^N$ ), in general, CR-functions on  $V$  do not extend to a wedge in the direction of each nontrivial value of the Levi form of  $M$  as was recently observed by Eastwood-Graham [EG99, EG01] already in the case when  $M$  is a hypersurface (see [EG99, Example 1.2] and Example 1.2 below). In this paper we propose an invariant geometric way

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of selecting those Levi form directions that are responsible for the extension, in particular, explaining the phenomenon discovered in [EG99, EG01].

The (extrinsic) Levi form of  $M$  at a point  $p$  can be seen as a hermitian form  $L = L_p: T_p^c M \times T_p^c M \rightarrow (T_p \mathbb{C}^N / T_p M) \otimes \mathbb{C}$ , where  $T_p^c M := T_p M \cap iT_p M$  is the complex tangent space. For every tangent vector  $w \in T_p^c M$  consider the opening angle  $0 \leq \gamma_w \leq 2\pi$  of the plane convex cone  $\mathbb{C}w \cap C_p V$ , where  $C_p V$  denotes the closed tangent cone to  $V$  at  $p \in \partial V$  (see §3). (We put  $\gamma_w = 0$  if  $\mathbb{C}w \cap C_p V$  contains no open cone and  $\gamma_w = 2\pi$  if  $\mathbb{C}w \cap C_p V = \mathbb{C}w$ .) Define now the *Levi*  $(\pi/2)$ -cone of  $V$  at  $p$  as the convex hull

$$\Gamma_p^{\pi/2} = \Gamma_p^{\pi/2}(V) := \text{conv} \{L(w, w) : \gamma_w > \pi/2\} \quad (1)$$

taken in the space  $T_p \mathbb{C}^N / T_p M$ . We refer to  $\gamma_w > \pi/2$  as the *angle condition*. Throughout this paper we write

$$\text{pr}: T_p \mathbb{C}^N \rightarrow T_p \mathbb{C}^N / T_p M$$

for the canonical projection. Our main result provides holomorphic extension of CR-functions on  $V$  in all additional directions of  $\Gamma_p^{\pi/2}(V)$ :

**Theorem 1.1.** *Let  $V$  be a wedge with edge  $E$  at  $p$  in a  $C^4$ -smooth submanifold  $M \subset \mathbb{C}^N$ , where  $E$  is a  $C^2$ -smooth generic submanifold of  $\mathbb{C}^N$ . Suppose that the interior of the cone  $\Gamma_p^{\pi/2} \subset T_p \mathbb{C}^N / T_p M$  is nonempty. Then all continuous CR-functions on  $V$  extend holomorphically to a wedge  $V'$  in  $\mathbb{C}^N$  with edge  $E$  at  $p$  such that*

- (i)  $\overline{V'}$  contains a neighborhood of  $p$  in  $V$ ;
- (ii)  $\text{pr}(C_p V') \supset \Gamma_p^{\pi/2}$ .

Moreover, the extension of each CR-function continuous up to  $E$  is also continuous up to  $E \cup V$ .

In particular, the wedge  $V'$  is contained in the polynomial, rational and holomorphic (with respect to any Stein neighborhood) convex hulls of  $V$ . The following example shows that  $\pi/2$  for the cone  $\Gamma_p^{\pi/2}$  cannot be replaced by any smaller angle:

*Example 1.2.* Consider the hypersurface

$$M := \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 : \text{Im } z_3 = (\text{Im } z_1)^2 - (\text{Im } z_2)^2\}$$

and the wedge  $V := \{z \in M : \text{Im } z_1 > |\text{Im } z_2|\}$  with edge  $E := \mathbb{R}^3$  at 0. Then for

$$w := (1, 1 - \varepsilon - i\sqrt{2\varepsilon}, 0) \in T_0^c M$$

with  $\varepsilon > 0$  arbitrarily small, the Levi form  $L(w, w)$  points down and the angle  $\gamma_w < \pi/2$  is arbitrarily close to  $\pi/2$ . However, any domain of holomorphic extension of CR-functions on  $V$  must be contained in the half-space  $\text{Im } z_3 > 0$ . Hence the Levi form of a vector  $w$  with  $\gamma_w$  arbitrarily close to  $\pi/2$ , may not be in general a direction of extension as in Theorem 1.1.

The assumption that the edge  $E$  is *generic* in Theorem 1.1 is also important and cannot be dropped as the following example shows:

*Example 1.3.* Consider the hypersurface  $M := \{z = (z_1, z_2) \in \mathbb{C}^2 : \operatorname{Im} z_2 = |z_1|^2\}$  and, for a real number  $1/2 < \alpha < 1$ , the wedge  $V := \{z \in M : \operatorname{Im} z_2 < \operatorname{Im} z_1^{1/\alpha}\}$  with edge  $E := \{z \in M : z_1 = 0\}$  at 0. Then, for  $w := (1, 0) \in T_0^c M$ , we have  $L_0(w, w) > 0$  and  $\gamma_w = \pi\alpha > \pi/2$ . However, the CR-function  $f(z_1, z_2) := (z_2 - z_1^{1/\alpha})^{-1}$  on  $V$  does not extend holomorphically to any wedge  $V'$  as in Theorem 1.1. (It can be shown that the polynomial, rational and holomorphic convex hulls of  $V$  do not contain any wedge in  $\mathbb{C}^2$  with edge  $E$ , see e.g. [ZZ01].)

If one restricts attention to vectors  $w \in T_p^c M \cap iT_p E$ , then the angle condition in (1) is automatically satisfied. Indeed, since  $E$  is generic at  $p$ , it is easy to see that the angle  $\gamma_w$  is always either  $2\pi$  or  $\pi$  depending on whether  $w$  belongs to  $T_p^c E$  or not. For Levi directions arising from vectors  $w$  of this kind, a similar extension result is due to [AH81, BP82] for  $\gamma_w = 2\pi$  and [BZ01] for  $\gamma_w = \pi$ . Another case has been treated in [EG99, EG01], where the cone  $C_p V$  contains zero directions of the Levi form with nonzero directions arbitrarily close. This situation can be also explained by Theorem 1.1, since the angle  $\gamma_w$  may be chosen arbitrarily close to  $\pi$ , in particular, greater than  $\pi/2$ . On the other hand, the following example shows that one may really need to consider  $\gamma_w$  arbitrarily close to  $\pi/2$  to get new directions of extension in  $T_p \mathbb{C}^N / T_p M$ :

*Example 1.4.* Consider the hypersurface

$$M := \{z = (z_1, z_2, z_3) \in \mathbb{C}^3 : \operatorname{Im} z_3 = |z_1|^2 + |z_2|^2 - (2 + \varepsilon)\operatorname{Im} z_1 \bar{z}_2\}$$

for a small  $\varepsilon > 0$ . Then the Levi form is negative only for vectors  $w \in T_0^c M$  whose direction is sufficiently close to that of  $w_0 = (-1 + i, 1 + i, 0)$ , for which  $L_0(w_0, w_0) = -2\varepsilon < 0$ . We choose  $V := \{z \in M : \operatorname{Im} z_1 > 0, \operatorname{Im} z_2 > 0\}$  so that  $\gamma_{w_0} = \pi/2$ . Then there exist vectors  $w$  arbitrarily close to  $w_0$  with  $\gamma_w > \pi/2$  in view of Lemma 2.1 in §2 and hence Theorem 1.1 yields extension to a wedge on the negative side of  $M$ . On the other hand, for any vector  $w$  with  $L_0(w, w) < 0$ , the angle  $\gamma_w$  must be close to  $\pi/2$ .

The previously known technique for proving wedge-extendibility of CR-functions is based on attaching smooth  $(C^{1,\beta})$  analytic discs to  $E \cup V$  which is possible in any direction  $w \in T_p^c M \cap iT_p E$  and then on filling a wedge by these discs. In the other directions  $w \in T_p^c M$  with  $\pi/2 < \gamma_w < \pi$  considered in this paper, such discs may not exist and hence other methods are required. Our approach here consists of two main steps. Using tools developed in [ZZ01], we first attach non-smooth analytic discs to  $V$  that will not fill a wedge anymore but rather a smaller region  $V'$  that we call  $\alpha$ -wedge, where some directions are bounded by  $\alpha$ th powers of others for some  $\alpha > 1/2$ . (Here  $\alpha$  is chosen such that  $\gamma_w = \alpha\pi$ .) Then

we attach analytic discs to properly chosen submanifolds  $\tilde{M}$  approximating  $M$  such that, over a certain region,  $\tilde{M}$  is contained in  $V'$ . For this step, the property  $\alpha > 1/2$  is crucial and guarantees that the directions of  $(1/\alpha)$ th powers are not affected by the Taylor expansion of (the defining functions of)  $M$ .

We conclude by giving an application of Theorem 1.1 to regularity and holomorphic extension of CR-functions.

**Corollary 1.5.** *Let  $M \subset \mathbb{C}^N$  be a  $C^4$ -smooth submanifold,  $V_1, \dots, V_n$  be a wedges in  $M$  with the same edge  $E$  at  $p$ , where  $E$  is a generic  $C^2$ -smooth submanifold. Suppose that*

- (i)  $C_p V_1 + \dots + C_p V_n = T_p M$ ;
- (ii)  $\Gamma_p^{\pi/2}(V_1) + \dots + \Gamma_p^{\pi/2}(V_n) = T_p \mathbb{C}^N / T_p M$ .

*Then every continuous function  $f$  on  $E$  that admits continuous CR-extension to each  $V_j$ , also extends holomorphically to a neighborhood of  $p$  in  $\mathbb{C}^N$ . In particular,  $f$  has the same smoothness as  $E$  and its extension to each  $V_j$  the same smoothness as  $M$ .*

In the case of two wedges  $V_1, V_2$  with opposite directions in a hypersurface  $M$  containing zero directions of the Levi form (of  $M$ ) having mixed signature similar results are due to Eastwood-Graham [EG99, EG01].

## 2. Equivalent version of the angle condition

The angle condition  $\gamma_w > \pi/2$  has a simple interpretation in terms of the directional cone of  $V$ . For any generic submanifold  $E \subset M \subset \mathbb{C}^N$  at a point  $p \in E$  we have the canonical identification

$$(iT_p E \cap T_p M) / T_p^c E \rightarrow T_p M / T_p E.$$

Then, if  $V \subset M$  is a wedge in  $M$  with edge  $E$  at  $p$ , the (closed) cone  $C_p V$  can be represented, modulo  $T_p^c E$ , by an open cone  $i\Sigma \subset iT_p E \cap T_p M$  with  $\text{span}(i\Sigma) \oplus T_p E = T_p M$  such that the interior of  $C_p V$  coincides with  $T_p E + i\Sigma$ . We have:

**Lemma 2.1.** *For  $w \in T_p^c M$ , the condition  $\gamma_w > \pi/2$  holds if and only if, for suitable  $e^{i\theta} \in S^1$ ,  $\tilde{w} \in T_p^c M$  arbitrarily close to  $w$  and for any decomposition*

$$e^{i\theta} \tilde{w} = a + ib \in T_p E + iT_p E \mod T_p^c E,$$

*one has  $b \pm a \in \Sigma \mod T_p^c E$ .*

*Proof.* By adding a suitable vector  $v \in \Sigma$  to  $w$ , we can assume that the plane  $\mathbb{C}w$  intersects the interior of  $C_p V$  in a plane cone of angle greater than  $\pi/2$ . The last condition holds if and only if, for some choice of  $e^{i\theta}$ ,  $a, b$  as above, one has

$$\Pi((1 - \tau)^\alpha(a + ib)) \in i\Sigma \text{ for all } \tau \in \Delta \quad (2)$$

for some  $\alpha > 1/2$ , where  $\Delta \subset \mathbb{C}$  denotes the unit disc and  $\Pi: T_p M \rightarrow \text{span}(i\Sigma)$  is the natural projection along  $T_p E$ . The inclusion in (2) is clearly equivalent to  $b + ta \in \Sigma$  for all  $t \in (-1 - \varepsilon, 1 + \varepsilon)$  for some  $\varepsilon > 0$  and hence to  $b \pm a \in \Sigma$ .  $\square$

A particular case when  $a = 0$  that is  $e^{i\theta}w = ib \in i\Sigma + T_p^c M$  or  $w \in T_p^c M$  was considered in [BZ01]. It is clear that this happens precisely when either  $\gamma_w = \pi$  or  $\gamma_w = 2\pi$ .

### 3. Preliminaries

We begin by defining open subsets in real manifolds having cone property. By a *cone*  $\Gamma$  in the euclidean space  $\mathbb{R}^m$  we always mean a subset invariant under multiplication by positive real numbers. A subcone  $\Gamma' \subset \Gamma$  is said to be *strictly smaller* if  $\overline{\Gamma'} \setminus \{0\}$  is contained in the interior of  $\Gamma$ .

**Definition 3.1.** An open subset  $V$  in a real  $C^1$ -smooth manifold  $M$  is said to have the cone property at a boundary point  $p \in \partial V$  with respect to an open cone  $\Gamma \subset T_p M$  if, for any strictly smaller subcone  $\Gamma' \subset \Gamma$  and some (and hence any) local coordinates on  $M$  in a neighborhood of  $p$ , one has  $x + y \in V$  for all  $x \in V$  and  $y \in \Gamma'$  sufficiently close to  $p$  and  $0$  respectively, where the sum is taken with respect to the given coordinates.

We also say that  $V$  has the cone property without specifying  $\Gamma$  if there exists  $\Gamma$  for which the cone property holds. It is clear from the definition that, if a given  $V \subset M$  has the cone property with respect to each of two cones  $\Gamma_1$  and  $\Gamma_2$ , then the cone property also holds for the sum  $\Gamma_1 + \Gamma_2$ . Furthermore, among all such cones there is unique maximal one, namely the sum of all of them that is automatically convex. We call it the *tangent cone* to  $V$  at  $p$  and denote by  $C_p V$ .

An important special case of a subset with cone property is a wedge. We call an open subset  $V \subset M$  a *wedge with edge*  $E$  at  $p$ , where  $E$  is a submanifold of  $M$  through  $p$ , if  $V$  has cone property at  $p \in \partial V$  and if  $E \subset \partial V$ . In this case the tangent cone  $C_p V$  is automatically invariant under addition of vectors in  $T_p E$  and can be canonically represented by a normal cone  $\Sigma \subset T_p M / T_p E$  that we call the *directional cone* of the wedge  $V$  at  $p$ .

Finally, recall that a real submanifold  $E \subset \mathbb{C}^N$  is called *generic at a point*  $p \in E$  if  $T_p E + iT_p E = \mathbb{C}^N$ . Generic submanifolds are precisely the noncharacteristic submanifolds for the system of Cauchy-Riemann equations. If  $M \subset \mathbb{C}^N$

is another submanifold containing  $E$  which is generic at  $p$ , then  $M$  is also generic at  $p$ . As usual continuous functions are called CR-functions if they are solutions of the tangential Cauchy-Riemann equations in distributional sense (see e.g. [BER99]).

#### 4. Attaching nonsmooth analytic discs

We discuss now about attaching analytic discs to a generic submanifold  $M$  of  $\mathbb{C}^N$ . The reader is referred to [B91, T96, BER99] for details. By an *analytic disc* in  $\mathbb{C}^N$  we mean a holomorphic mapping  $A$  from  $\Delta$ , the unit disc in  $\mathbb{C}$ , into  $\mathbb{C}^N$  which extends at least continuously to  $\bar{\Delta}$ ; we still denote by  $A$  the image  $A(\Delta)$ . We say that  $A$  is *attached* to  $M$  if  $A(\partial\Delta) \subset M$ . Usually one considers smooth  $(C^{k,\beta})$  with  $k \geq 1$  discs. (The reason of the fractional regularity is due to the continuity of the Hilbert transform in the corresponding classes of functions.) We will also make an extensive use of the main technical tools developed in [ZZ01] for attaching analytic discs that are  $C^1$  except at  $\tau = 1$ , where they are only Hölder-continuous. For  $\frac{1}{2} < \alpha < 1$ , we put

$$\beta := 2\alpha - 1.$$

For any real  $\delta$ , consider the principal branch of  $(1 - \tau)^\delta$  on  $\Delta$  that is real on the interval  $(0, 1)$ ; note that  $(1 - \tau)^{2\alpha} \in C^{1,\beta}$ . As in [ZZ01], define  $\mathcal{P}^\alpha(\partial\Delta) \subset C^\alpha(\partial\Delta)$  to be the subspace of all linear combinations of  $(1 - \tau)^\alpha$ ,  $(1 - \bar{\tau})^\alpha$  and functions in  $C^{1,\beta}(\partial\Delta)$  and  $\mathcal{P}^\alpha(\bar{\Delta}) \subset C^\alpha(\bar{\Delta})$  to be the subspace of linear combinations of  $(1 - \tau)^\alpha$ ,  $(1 - \bar{\tau})^\alpha$  and functions in  $C^{1,\beta}(\bar{\Delta})$ . These subspaces become Banach spaces with the norm of a disc being the sum of the  $C^\alpha$ -norm of its nonsmooth term and the  $C^{1,\beta}$ -norm of the remainder. It is easy to see that the Hilbert transform is a bounded operator on  $\mathcal{P}^\alpha(\bar{\Delta})$ . We choose coordinates  $(z, w) \in \mathbb{C}^l \times \mathbb{C}^n = \mathbb{C}^N$ ,  $z = x + iy$ , where  $M$  is defined by  $y = h(x, w)$  for a vector valued function  $h = (h_1, \dots, h_l)$  with  $h(0) = 0$ ,  $h'(0) = 0$ . It is well-known (cf. [B91]) that, given a sufficiently small holomorphic function  $w(\tau)$  in  $\Delta$  that is  $C^\alpha$  in  $\bar{\Delta}$  and a sufficiently small vector  $x \in \mathbb{R}^l$ , there exists a (unique) small analytic disc  $A(\tau) = (z(\tau), w(\tau))$  attached to  $M$  in the same class with prescribed small value  $x(1) = x$ . It is shown in [ZZ01] that the same statement also holds in the class  $\mathcal{P}^\alpha$  defined above and that the obtained disc  $A$  depends smoothly on  $w(\cdot)$  and  $x$ . More precisely, one has:

**Proposition 4.1 ([ZZ01]).** *Let  $M$  be  $C^{k+2}$ -smooth ( $k \geq 1$ ) and fix  $\alpha > 1/2$ . Then for every sufficiently small  $x \in \mathbb{R}^l$  and holomorphic function  $w(\cdot) \in \mathcal{P}^\alpha(\bar{\Delta})$  there is an unique sufficiently small analytic disc  $A = (z(\cdot), w(\cdot)) \in \mathcal{P}^\alpha$  with  $x(1) = x$  attached to  $M$  that depends in a  $C^k$  fashion on  $w \in \mathcal{P}^\alpha$  and  $x \in \mathbb{R}^l$ .*

It is further shown in [ZZ01] that, if  $A(1) = 0$  and if the coordinates  $(z, w)$  are chosen as above, the component  $z(\tau)$  is in fact in  $C^{1,\beta}$ .

## 5. Filling $\alpha$ -wedges by nonsmooth analytic discs

Our first main step in proving Theorem 1.1 will be to obtain a weaker statement about the extension to a region smaller than a wedge. More precisely, given submanifolds  $M \subset M' \subset \mathbb{C}^N$  and an open subset, define an  $\alpha$ -wedge in  $M'$  over  $V$  at a point  $p \in \partial V$  in the additional directions of an open convex cone  $\Gamma \subset T_p M' / T_p M$  to be the open subset of  $M'$  of the form

$$V' = \{z \in W : \text{dist}(z, V) < c \text{dist}(z, \partial V)^{1/\alpha}\}, \quad (3)$$

where  $W$  is a wedge in  $M'$  with edge  $M$  at  $p$  and  $c$  is a positive constant. We say that an  $\alpha$ -wedge over  $V$  is of class  $C^{1,\delta}$  if the submanifold  $M'$  can be chosen to be of class  $C^{1,\delta}$ .

The following statement is the main technical tool for the proof of Theorem 1.1:

**Theorem 5.1.** *Let  $M \subset \mathbb{C}^N$  be a generic  $C^4$ -smooth submanifold and  $V \subset M$  be a domain with cone property at a point  $p \in \partial V$ . Fix  $\alpha > 1/2$  and  $w \in T_p^c M$  such that*

$$\gamma_w > \alpha\pi, \quad L_p(w, w) \neq 0. \quad (4)$$

*Then analytic discs in  $\mathbb{C}^N$  attached to  $V$  fill an  $\alpha$ -wedge over  $V$  at  $p$  of class  $C^{1,\delta}$  for some  $0 < \delta < 1$  with one-dimensional additional direction arbitrarily close to that of  $L_p(w, w)$ .*

Here the new  $\alpha$ -wedge is understood in a submanifold  $M'$  with boundary  $M$  at  $p$ . When  $V$  is a wedge with generic edge  $E$ , then the arguments of Baouendi-Treves [BT81] can be used to show that continuous CR-functions on a neighborhood of  $p$  in  $V$  can be uniformly approximated on compacta by holomorphic polynomials. Hence, replacing  $V$  by such a neighborhood, we obtain from Theorem 5.1:

**Corollary 5.2.** *Let  $V$  be a wedge in a  $C^4$ -smooth submanifold  $M \subset \mathbb{C}^N$  with generic edge  $E$  at a point  $p$ . Then all continuous CR-functions on  $V$  admit CR-extension to an  $\alpha$ -wedge over  $V$  at  $p$  of class  $C^{1,\delta}$  for some  $0 < \delta < 1$  with one-dimensional additional direction arbitrarily close to that of  $L_p(w, w)$ .*

Indeed, given a sequence of holomorphic polynomials converging uniformly to a continuous CR-function on  $V$  on compacta, it also converges uniformly on the union of discs by the maximum principle. Hence, if the discs fill an  $\alpha$ -wedge over  $V$ , the sequence of polynomials converges there to a continuous CR-function.

**Remark 5.3.** If a CR-function is of class  $C^\kappa$  for some  $1 \leq \kappa \leq \infty$ , the sequence of polynomials can be chosen to converge in the  $C^\kappa$ -norm and therefore the CR-extension is also  $C^\kappa$ .

*Proof of Theorem 5.1.* Choose coordinates  $(z, w) \in \mathbb{C}^l \times \mathbb{C}^n$ ,  $z = x + iy$ , with  $p = 0$  in which  $M$  is defined by  $y = h(x, w)$ , where  $h := (h_1, \dots, h_l)$  has no harmonic terms of order 2. Let  $w = w_0 \in T_p^c M$  satisfy (4). By multiplying  $w_0$  with a nonzero complex number we may assume that  $(1 - \tau)^{\alpha'} w_0 \in \Gamma$  for some  $\alpha' > \alpha$  and all  $\tau \in \Delta$ , where  $\Gamma$  is a suitable strictly smaller subcone of the interior of  $C_p V$ . For a small real parameter  $\eta > 0$  set

$$w(\tau) = w_\eta(\tau) := \eta(1 - \tau)^\alpha w_0.$$

We attach discs  $A(\tau) = A_\eta(\tau)$  to  $M$  with “ $w$ -components”  $w(\tau)$  (the above choice of  $w_0$  will guarantee that our discs will be automatically attached to  $V$ ). We write

$$z(\tau) = u(\tau) + iv(\tau), \quad \tau = re^{i\theta} \in \overline{\Delta},$$

and solve the Bishop’s equation

$$u(\tau) = -T_1 h(u(\tau), w(\tau)), \quad \tau = e^{i\theta} \in \partial\Delta,$$

where  $T_1$  is the Hilbert transform on  $\partial\Delta$  normalized by the condition  $T_1(\cdot)|_{\tau=1} = 0$ . By Proposition 4.1, for small  $\eta$ , this equation has solution  $u(\cdot)$  in  $\mathcal{P}^\alpha$  with  $u(1) = 0$ . It is proved in [ZZ01] that  $u(\cdot)$  is in  $C^{1,\beta}$  and moreover  $\eta \mapsto v_\eta$ ,  $\mathbb{R} \rightarrow C^{1,\beta}$  is of class  $C^k$ . Then also  $v := T_1 u \in C^{1,\beta}$  and  $\eta \mapsto v_\eta$  is of class  $C^k$ . In particular, for the radial derivative  $\partial_r v$ , we have

$$\partial_r v = \partial_r v|_{\eta=0} + (\partial_r \dot{v}|_{\eta=0})\eta + (\partial_r \ddot{v}|_{\eta=0})\eta^2 + o(\eta^2), \quad \eta \rightarrow 0, \quad (5)$$

where the dots stand for the derivatives in  $\eta$ . Since  $h(0) = 0$  and  $h'(0) = 0$ , we have  $v|_{\eta=0} \equiv 0$  and  $\dot{v}|_{\eta=0} \equiv 0$ . Hence also  $\dot{u}|_{\eta=0} \equiv 0$  because  $\dot{u}$  is related to  $\dot{v}$  by the Hilbert transform. As for  $\ddot{v}$ , note that double differentiation in  $\eta$  of  $v(\cdot) = h(u(\cdot), w(\cdot))$  along  $\partial\Delta$  yields

$$\ddot{v}|_{\eta=0} = L_p(w_0, w_0)|1 - \tau|^{2\alpha}, \quad \tau \in \partial\Delta. \quad (6)$$

Note that for  $\tau = e^{i\theta}$ , we have  $|1 - \tau|^{2\alpha} = |\theta|^{2\alpha} + o(|\theta|^{2\alpha}) \in C^{1,\beta}(\partial\Delta)$ . Hence the harmonic extension of  $|1 - \tau|^{2\alpha}$  from  $\partial\Delta$  to  $\Delta$  is also of class  $C^{1,\beta}(\partial\Delta)$ . By the Hopf Lemma, the radial derivative of this harmonic extension at  $\tau = 1$  is negative and we have by (6),

$$\partial_r \ddot{v}|_{\tau=1, \eta=0} = cL_p(w_0, w_0), \quad c < 0.$$

Hence it follows from (5) that we can fix sufficiently small  $\eta > 0$  such that  $\partial_r v \notin T_p M$  and such that  $\partial_r v$  points to a direction arbitrarily close to that of  $L_p(w_0, w_0)$ . We write  $A_p$  for the corresponding analytic disc. The above construction can be used to obtain a family of attached analytic discs  $A_q$  for  $q \in \overline{V}$  near  $p$  having the same “ $w$ -component” and such that  $A_q(1) = q$ . The conclusion about filling an  $\alpha$ -wedge over  $V$  follows now from a result of [ZZ01] that we state here for the reader’s convenience:

**Lemma 5.4.** ([ZZ01]) *Let  $V \subset \mathbb{R}^l \times \{0\} \subset \mathbb{R}^l \times \mathbb{R}^{m-l}$  be an open subset with Lipschitz boundary at  $0 \in \partial V$ . Let  $A: [0, 1] \times \bar{V} \rightarrow \mathbb{R}^m$  be a map of the form*

$$\varphi(t, q) = q + t^\alpha a(q) + b(t, q), \quad 0 < \alpha < 1,$$

*with  $a(\cdot)$  and  $b(\cdot, q)$ ,  $q \in \bar{V}$ , being of class  $C^{1,\gamma}$  for some  $0 < \gamma \leq \frac{1}{\alpha} - 1$  such that  $a(q) \in C_0 V \times \{0\}$ ,  $b(0, q) \equiv 0$  and  $\partial_t b(0, 0) \notin \mathbb{R}^l \times \{0\}$ . Assume that the induced map  $q \mapsto b(\cdot, q)$  between  $\bar{V}$  and  $C^{1,\gamma}$  is also of class  $C^{1,\gamma}$ . Then there exists  $\varepsilon > 0$ , a neighborhood  $U$  of  $0$  in  $\mathbb{R}^l \times \{0\}$  and a submanifold  $M'$  of class  $C^{1,\gamma^2}$  with boundary  $\mathbb{R}^l \times \{0\}$  at  $0$  in the direction of  $\partial_t b(0, 0)$  such that  $A$  defines a homeomorphism between  $(0, \varepsilon) \times (V \cap U)$  and an  $\alpha$ -wedge over  $V$  at  $0$  in  $M'$ .*

□

If (4) holds for several vectors  $w_1, \dots, w_l \in T_p^c M$  such that  $L_p(w_1, w_1), \dots, L_p(w_l, w_l)$  is a basis in  $T_p \mathbb{C}^N / T_p M$ , then CR-functions on  $V$  extend to a full-dimensional  $\alpha$ -wedge over  $V$  whose additional directions can be chosen to be in any cone strictly smaller than the convex span of the above basis. This result will follow from the proof of the “edge-of-the-wedge” theorem of Ajrapetian and Henkin [AH81] in the version of [A87] (see also [T96]), where attached analytic discs have been used. More precisely, for any positive real number  $\alpha$  with  $\alpha > \frac{1}{2}$ , define the Levi  $\alpha\pi$ -cone  $\Gamma_p^{\alpha\pi} \subset T_p \mathbb{C}^N / T_p M$  to be the convex hull

$$\Gamma_p^{\alpha\pi} := \text{conv}\{L_p(w, w) : w \in T_p^c M, \gamma_w > \alpha\pi\}.$$

**Corollary 5.5.** *In the setting of Theorem 5.1, assume that  $\Gamma_p^{\alpha\pi}$  has a nonempty interior in  $T_p \mathbb{C}^N / T_p M$ . Then for any strictly smaller subcone  $\Gamma' \subset \Gamma_p^{\alpha\pi}$ , analytic discs attached to  $V$  fill an  $\alpha$ -wedge  $V'$  over  $V$  at  $p$  with additional directions of  $\Gamma'$ . In particular, all continuous CR-functions on  $V$  that are approximated by polynomials on compacta admit holomorphic extension to  $V'$ .*

*Remark 5.6.* If the boundary  $\partial V$  does not contain a generic submanifold  $E$ , the classical arguments of [BT81] cannot be applied to show that any continuous CR-function on  $V$  is approximated by polynomials on compacta, even after replacing  $V$  by a smaller neighborhood of  $p$  in  $V$ . Nevertheless it is shown in [ZZ01] by a different construction that holomorphic extension of CR-functions to an  $\alpha$ -wedge can be obtained also in this case.

*Proof of Corollary 5.5.* We write  $\Gamma_1 \subset\subset \Gamma_2$  if  $\Gamma_1$  is strictly smaller than  $\Gamma_2$ . Given  $\Gamma' \subset\subset \Gamma_p^{\alpha\pi}$  as above, consider a polyhedral approximation of  $\Gamma'$  in  $\Gamma_p^{\alpha\pi}$ :

$$\Gamma' \subset\subset \text{conv}\{L_p(w_1, w_1), \dots, L_p(w_s, w_s)\} \subset \Gamma_p^{\alpha\pi}. \quad (7)$$

By Corollary 5.2, all (continuous) CR-functions on  $V$  admit CR-extensions to  $\alpha$ -wedges  $\tilde{V}_1, \dots, \tilde{V}_s$  over  $V$  at  $p$  of class  $C^{1,\delta}$  with additional directions of

vectors  $v_1, \dots, v_s$  arbitrarily close to those of  $L_p(w_1, w_1), \dots, L_p(w_s, w_s)$  respectively. Denote by  $W_1, \dots, W_s$  the corresponding wedges with edge  $M$  at  $p$  as in (3). By (7), we can assume  $\Gamma' \subset \subset \text{conv}\{v_1, \dots, v_s\}$ . Then there exists a constant  $C > 0$  such that for any  $z_0 \in V$  and  $1 \leq j \leq s$ ,

$$\{z \in W_j : \|z - z_0\| < c \text{dist}(z_0, \partial V)^{1/\alpha}\} \subset \tilde{V}_j. \quad (8)$$

By repeating the arguments of the proof of [T96, Theorem 4.1] one can show that all continuous CR-functions on  $V$  admit holomorphic extensions to a subset

$$\{z \in B(p) + \Gamma' : \|z - z_0\| < c' \text{dist}(z_0, \partial V)^{1/\alpha}\}, \quad (9)$$

where  $B(p)$  is a neighborhood of  $p$  in  $M$  and  $c' > 0$  is a constant, both chosen independently of  $z_0$  (we have identified  $\Gamma'$  with a cone in  $\mathbb{C}^N$  in the normal direction to  $T_p M$ ). Such a choice can be obtained by rescaling the submanifold in (8) and using the uniformity of the construction of Tumanov [T96] with respect to the  $C^{1,\delta}$ -norms of the submanifolds and using his regularity results [T93] for parameter dependence of solutions of Bishop's equation. It follows now from (3) that the subset (9) contains an  $\alpha$ -wedge over  $V$  with additional directions of  $\Gamma'$ .  $\square$

## 6. Passing from $\alpha$ -wedges to wedges

In this section we will assume that  $V \subset M$  is a wedge with generic edge  $E$  at  $p$ . We denote by  $(C_p V)^0$  the interior in  $T_p M$  of the tangent cone  $C_p V$  of  $V$  at  $p$ . As before we fix  $\alpha > 1/2$ .

**Proposition 6.1.** *Let  $V$  be a wedge in  $M$  with generic edge  $E$  at  $p$  and  $V'$  be an  $\alpha$ -wedge over  $V$  at  $p$  with one-dimensional additional direction. Then the union of analytic discs attached to  $V \cup V'$  contains a wedge  $V''$  with edge  $E$  at  $p$  in a submanifold  $M'' \subset \mathbb{C}^N$  with  $\dim M'' = \dim M + 1$  such that  $\bar{V}''$  contains a neighborhood of  $p$  in  $V$  and  $\text{pr}(C_p V'')$  is arbitrarily close to the additional direction of  $V'$ . In particular, all continuous functions on  $V$  admitting CR-extension to  $V'$  are also CR-extendible to a wedge with edge  $E$  at  $p$  as above.*

*Proof.* Since  $E$  is generic, by shrinking  $V$  and  $V'$ , if necessary, we may assume that continuous functions on  $V \cup V'$  that are CR on  $V'$  are uniformly approximated by holomorphic polynomials on compacta. By choosing suitable holomorphic coordinates  $z = x + iy \in \mathbb{C}^N$  in a neighborhood of  $p$  we may assume that  $p = 0$ ,

$$T_0 E \subset \{y_{N-1} = y_N = 0\}, \quad T_0 M \subset \{y_N = 0\}, \quad (0, \dots, 0, i, 0) \in \text{int}(C_p V)$$

and the additional direction of  $V'$  is that of  $(0, \dots, 0, i)$ . We write  $z = (z'', z_{N-1}, z_N) \in \mathbb{C}^N$ ,  $z' = (z'', z_{N-1}) \in \mathbb{C}^{N-1}$  and denote by  $\sigma : B(p) \rightarrow M$  any transversal projection from a neighborhood  $B(p)$  of  $p$  in  $\mathbb{C}^N$  to  $M$ . In the

sequel we write  $c_j$  ( $j = 1, 2, \dots$ ) for positive constants that may change during the proof. Since  $M$  is of class  $C^2$ , we have

$$y_N > -c_1 \|(z', x_N)\|^2, \quad z = (z', x_N + iy_N) \in M$$

with  $c_1$  suitably chosen. Let  $W$  be a wedge with edge  $M$  at  $p$  as in (3). Then there exist  $c_1, c_2 > 0$  such that

$$\{z \in W : \sigma(z) \in V, \quad c_1 \|(z', x_N)\|^2 < y_N < -c_1 \|(z', x_N)\|^2 + c_2 \text{dist}(z, \partial V)^{1/\alpha}\} \subset V'. \quad (10)$$

Let further  $\tilde{V} \subset V$  be a wedge with edge  $E$  at  $p$  whose directional cone at  $p$  is a sufficiently small cone around  $[(0, \dots, 0, 1, 0)] \in T_p M / T_p E$ . Then for  $z$  near  $p$  with  $\sigma(z) \in \tilde{V}$ , we have

$$\text{dist}(z, \partial V) > c_1 |y_{N-1}| - c_2 \|(z'', x_{N-1}, x_N)\|^2 \quad (11)$$

for suitable constants  $c_1, c_2 > 0$ , where we have used that  $E$  is of class  $C^2$ . From (10) and (11) we conclude

$$\{z \in W : \sigma(z) \in \tilde{V}, \quad c_1 \|(z', x_N)\|^2 < y_N < -c_1 \|(z'', x_{N-1}, x_N)\|^2 + c_2 |y_{N-1}|^{1/\alpha}\} \subset V'. \quad (12)$$

We shall now define a deformation  $\tilde{M}$  of  $M$  whose part will enter the region in the left-hand side of (12). It is easy to see from the construction that, if  $c_3$  is sufficiently large,

$$\tilde{M} := \{z \in \overline{W} : y_N = c_3 \|(z', x_N)\|^2\}$$

is a submanifold of  $\mathbb{C}^N$  near  $p$  of the same dimension as  $M$ . Furthermore we can shrink  $\tilde{M}$  around  $p$  and choose a sufficiently large constant  $c_4$  for which (12) implies the inclusion

$$\{z \in \tilde{M} : \sigma(z) \in \tilde{V}, \quad y_{N-1} > c_4 \|(z'', x_{N-1}, x_N)\|^{2\alpha}\} \subset V'. \quad (13)$$

We now construct an analytic disc attached to  $\tilde{M} \cap V'$  that passes through 0 and is not tangent to  $M$  there. Consider a holomorphic function  $z_{N-1}(\cdot)$  on the unit disc  $\Delta$  that is  $C^{1,\beta}$  ( $\beta = 2\alpha - 1$ ) up to the boundary with  $z_{N-1}(1) = 0$  and satisfying

$$y_{N-1}(\tau) \geq c_5 |x_{N-1}(\tau)|^{2\alpha}, \quad \tau \in \overline{\Delta}. \quad (14)$$

By solving the Bishop's equation for sufficiently small  $z_{N-1}$  (in the  $C^{1,\beta}$  norm), we can find a small analytic disc  $z(\tau)$  with  $z(1) = 0$  and the nontrivial tangential derivative  $\partial_\theta z(1) \neq 0$ , attached to  $\tilde{M}$  with the component  $z_{N-1}(\tau)$  prescribed above such that the derivatives  $\partial_\theta z(\tau)$ ,  $\tau \in \Delta$ , are arbitrarily close to the plane  $\{(0, \dots, 0)\} \times \mathbb{C} \times \{0\} \subset T_p M$ . Then we can achieve  $\sigma(z(\tau)) \in \tilde{V}$  for all  $z \in \Delta$ .

Furthermore, since  $\|(z''(\tau), z_N(\tau))\| \leq c_6|1 - \tau|$  with  $c_6$  arbitrarily small and  $|x_{N-1}(\tau)| \geq c_7|1 - \tau|$  for some  $c_7$  and all  $\tau \in \partial\Delta$  in a suitable neighborhood  $U$  of 1, we conclude

$$|x_{N-1}(\tau)|^{2\alpha} \geq \frac{1}{2}|x_{N-1}(\tau)|^{2\alpha} + c_8\|(z''(\tau), x_N(\tau))\|^{2\alpha}$$

with  $c_8$  arbitrarily large. Therefore  $z = z(\tau)$  is contained in the left-hand side of (14) for  $z \in \partial\Delta \cap U \setminus \{1\}$ . On the other hand, it is clear from the construction that  $z(\tau)$  can be chosen to be contained in the left-hand side of (14) also for  $\tau \in \partial\Delta \setminus U$ . It follows that  $z(\tau) \in V'$  for all  $\tau \in \partial\Delta \setminus \{1\}$ . Furthermore, due to the construction of  $\tilde{M}$  and the Hopf lemma, the radial derivative  $\partial_r z(1)$  is arbitrarily close to the subspace  $\{(0, \dots, 0)\} \times \mathbb{C}^2 \subset \mathbb{C}^{N-2} \times \mathbb{C}^2$  and  $\partial_r z(1) \notin T_p M$ . Hence sufficiently small deformations  $\tilde{z}(\cdot)$  of  $z(\cdot)$  with  $\tilde{z}(1) \in V$  fill a wedge  $V''$  satisfying the required conclusion.  $\square$

*Proof of Theorem 1.1.* Fix  $w \in T_p^c M$  with  $\gamma_w > \pi/2$ . We may assume that continuous CR-functions on  $V$  are uniformly approximated by holomorphic polynomials on compacta. By Corollary 5.2, all continuous CR-functions on  $V$  extend holomorphically to an  $\alpha$ -wedge  $V'$  over  $V$  at  $p$  with one-dimensional additional direction arbitrarily close to that of  $L_p(w, w)$ . Then we conclude from Proposition 6.1 that CR-functions on  $V$  also extend to wedge  $V''$  in a submanifold  $M''$  with  $\dim M'' = \dim M + 1$  satisfying the conclusion of the proposition. In this way we obtain CR-extension to a  $(\dim M + 1)$ -dimensional wedge  $V''$  with  $\text{pr}(C_p V'')$  arbitrarily close to any given Levi form value  $L_p(w, w)$  for  $\gamma_w > \pi/2$ .

If a CR-function on  $V$  is continuous up to  $E$ , it is approximated by polynomials uniformly up to  $E$  and hence the extension is continuous up to  $E \cup V$ . For these function, their holomorphic extension to a full-dimensional wedge  $V'$  with  $\text{pr}(C_p V') \supset \Gamma_p^{\pi/2}$  can be obtained from the mentioned above proof of a strong version of the edge-of-the-wedge theorem given by Tumanov [T93, T96].

If a CR-function is not known to be continuous up to  $E$ , the edge  $E$  can be “pushed” inside  $V$  reducing the extension problem to the above situation. Then the conclusion follows from the parameter version of the edge-of-the-wedge theorem following the arguments of [T93, T96].  $\square$

*Proof of Corollary 1.5.* If each of the cones  $\Gamma_p^{\pi/2}(V_1), \dots, \Gamma_p^{\pi/2}(V_n)$  has non-empty interior, the conclusion follows directly from Theorem 1.1 and from the edge-of-the-wedge theorem [AH81]. In general, the proof uses Corollary 5.2 and Proposition 6.1 as above to obtain CR-extension to lower-dimensional wedges and then again the edge-of-the-wedge theorem. For the last passage one needs the stronger version of the edge-of-the-wedge theorem assuming only  $C^{1,\delta}$ -regularity of the submanifolds that follows from [T93, T96] (see also [A87]).  $\square$

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