REMARKS ON THE RIGIDITY OF CR-MANIFOLDS

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Abstract. We propose a procedure to construct new smooth CR-manifolds whose local stability groups, equipped with their natural topologies, are subgroups of certain (finite-dimensional) Lie groups but not Lie groups themselves.

1. Introduction

Given a germ $(M, p)$ of a real submanifold of $\mathbb{C}^n$, its basic invariant is the local stability group $\text{Aut}(M, p)$, i.e. the group of all germs at $p$ of local biholomorphic maps of $\mathbb{C}^n$ fixing $p$ and preserving the germ $(M, p)$. By the work of several authors [CM74, BER97, Z97, ELZ03, LM05] it is known that this group is a (finite-dimensional) Lie group (in the natural inductive limit topology) for germs of real-analytic submanifolds satisfying certain nondegeneracy conditions, e.g. those having nondegenerate Levi form. On the other hand, in the absence of the nondegeneracy conditions, the group $\text{Aut}(M, p)$ can possibly be infinite-dimensional (in the sense that it contains Lie groups of arbitrarily large dimension). (E.g. the local stability group of $(\mathbb{R}, 0)$ in $\mathbb{C}$ consists of all convergent power series with real coefficients.) Furthermore, recent results in [BRWZ04] show that a similar principle also holds for global CR-automorphisms, both real-analytic and smooth.

One purpose of this paper is to show that, in contrast with the behaviour mentioned above, a similar alternative does not anymore hold for the local stability group of a smooth real submanifold. In particular, we show that, for any $n \geq 2$, there exists a germ $(M, p)$ of a smooth strongly pseudoconvex hypersurface in $\mathbb{C}^n$ with $\text{Aut}(M, p)$ being (topologically) isomorphic to a countable dense subgroup of the circle $S^1 \subset \mathbb{C}$ and hence not being a Lie group. In fact, $\text{Aut}(M, p)$ can be arranged to be isomorphic to any increasing countable union of finite subgroups of $S^1$, for instance, to the subgroup

$$\{e^{2\pi i \frac{l}{m}} : l, m \in \mathbb{N}\} \subset S^1.$$  \hfill (1.1)

Furthermore, our construction yields similar properties also for the (generally larger) local CR stability group $\text{Aut}_{\text{CR}}(M, p)$, consisting of all germs at a point $p$ of smooth CR-automorphisms of $M$ fixing $p$. Recall that a germ of a smooth transformation $\varphi : (M, p) \to (M, p)$ is a CR-automorphism if it preserves the subbundle $T^c M$ and and the restriction
of its differential $d\varphi|_{T^*M}$ is $\mathbb{C}$-linear, where

$$T^*M := TM \cap iTM.$$ 

Another purpose of this paper is to provide a general construction of new smooth generic submanifolds with certain prescribed local CR stability groups (recall that a real submanifold $M \subset \mathbb{C}^n$ is generic if $T_qM + iT_qM = T_q\mathbb{C}^n$ for all $q \in M$). More precisely, we show the following:

**Theorem 1.1.** Let $(M, p)$ be a germ of a smooth generic submanifold in $\mathbb{C}^n$ of positive codimension and of finite type and assume that it is invariant under an increasing countable union $G$ of finite subgroups of $\text{Aut}(\mathbb{C}^n, p)$. Then there exists a $G$-invariant germ of another smooth generic submanifold $(\tilde{M}, p)$ of the same dimension as $(M, p)$, which is tangent to $(M, p)$ of infinite order and has the following properties:

(i) $\text{Aut}(\tilde{M}, p) = G$;

(ii) $\text{Aut}_{CR}(\tilde{M}, p) = \{g|_M : g \in G\}$.

We use here the notion of finite type due to Kohn [K72] and Bloom-Graham [BG77]: a germ $(M, p)$ is of finite type, if all germs at $p$ of smooth real vector fields on $M$ tangent to $T^*M$ span together with their iterated commutators the full tangent space $T_pM$.

We now illustrate Theorem 1.1 by an example, where it can be applied.

**Example 1.2.** Consider a real hypersurface $M \subset \mathbb{C}^{n+1}, n \geq 1$, given in coordinates $(z_1, \ldots, z_n, w) \in \mathbb{C}^{n+1}$ by

$$\text{Im} w = \varphi(|z_1|^2, z_2, \ldots, z_n, \bar{z}_2, \ldots, \bar{z}_n, \text{Re} w),$$

where $\varphi$ is any smooth function such that $0 \in M$ and $(M, 0)$ is of finite type. Then $(M, 0)$ is clearly invariant under the rotation group consisting of all transformations $(z_1, z_2, \ldots, z_n, w) \mapsto (e^{2\pi i \theta} z_1, z_2, \ldots, z_n, w)$ for all real $\theta$. Now we can take the subgroup $G$ consisting of all these transformations corresponding to $\theta = l/2m$ with $l, m$ being positive integers. Then $G$ clearly satisfies the assumptions of Theorem 1.1. We then conclude that there exists a new real submanifold $\tilde{M} \subset \mathbb{C}^{n+1}$ such that both $\text{Aut}(\tilde{M}, p)$ and $\text{Aut}_{CR}(\tilde{M}, p)$ are (topologically) isomorphic to $G$, which is a topological subgroup of $S^1$ but is not itself a Lie group. Similar examples can be obtained for other $S^1$-actions or $S^1 \times S^1$-actions or actions by more general compact groups leaving $(M, p)$ invariant, where $M$ can also be of any codimension.

As a remarkable consequence of Theorem 1.1 and the mentioned results [BER97, Z97], our construction provides germs of smooth generic submanifolds (even of strongly pseudoconvex hypersurfaces) that are not CR-equivalent to any germ of any real-analytic CR-manifold. Recall that $(M, p)$ is called finitely nondegenerate if

$$\text{span}_{\mathbb{C}} \{L_1 \ldots L_k \rho_j^p(p) : k \geq 0, 1 \leq j \leq d\} = \mathbb{C}^n,$$  

(1.2)
where \( \rho = (\rho^1, \ldots, \rho^d) \) is a defining function of \( M \) near \( p \) with \( \partial \rho^1 \wedge \cdots \wedge \partial \rho^d \neq 0 \), \( \rho_z^i \) denotes the complex gradient of \( \rho^i \) in \( \mathbb{C}^n \) and the span is taken over all collections of germs at \( p \) of smooth \((0, 1)\) vector fields \( L_1 \ldots L_k \) on \( M \). We have:

**Corollary 1.3.** For any germ \((M, p)\) of a smooth generic submanifold in \( \mathbb{C}^n \), which is finitely nondegenerate and of finite type, and which is invariant under the group \( S^1 := \{ z \in \mathbb{C} : |z| = 1 \} \) acting on \( \mathbb{C}^n \) by multiplication, there exists a germ of another smooth generic submanifold \((\hat{M}, p)\), tangent to \((M, p)\) of infinite order, which is not CR-equivalent to any germ of a real-analytic generic submanifold of \( \mathbb{C}^n \).

Indeed, since \( S^1 \) has many dense subgroups \( G \) satisfying the assumptions of Theorem 1.1 (e.g. the subgroup in (1.1)), Theorem 1.1 yields a germ \((\hat{M}, p)\) whose local CR stability group is isomorphic to \( G \) and hence is not topologically isomorphic to any Lie group (and not locally compact). On the other hand, the local CR stability group of any real-analytic generic submanifold of \( \mathbb{C}^n \), which is CR-equivalent to \((\hat{M}, p)\) (and hence is also finitely nondegenerate and of finite type), is known to be always a Lie group (see [BER97] for hypersurfaces and [Z97] for higher codimension). Since the local CR stability group is a CR-invariant, \((\hat{M}, p)\) cannot be CR-equivalent to any germ of a real-analytic generic submanifold of \( \mathbb{C}^n \).

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### 2. Jet spaces and jet groups

Here we recall the jet terminology and introduce the notation that will be used throughout the paper. Recall that, given two complex manifolds \( X \) and \( X' \) and an integer \( k \geq 0 \), a \( k \)-jet of a holomorphic map is an equivalence class of holomorphic maps from open neighborhoods of \( x \) in \( X \) into \( X' \) with fixed partial derivatives at \( x \) up to order \( k \). Denote by \( J^k_x(X, X') \) the set of all such \( k \)-jets. The union \( J^k(X, X') := \bigcup_{x \in X} J^k_x(X, X') \) carries a natural fiber bundle structure over \( X \). For a holomorphic map \( f \) from a neighborhood of \( x \) in \( X \) into \( X' \), denote by \( j^k_x f \in J^k_x(X, X') \) the corresponding \( k \)-jet. In local coordinates, \( j^k_x f \) can be represented by the coordinates of the reference point \( x \) and all partial derivatives of \( f \) at \( x \) up to order \( k \). If \( X' \) and \( X' \) are smooth algebraic varieties, \( J^k_x(X, X') \) and \( J^k(X, X') \) are also of this type. We also denote by \( J^k_{x,x'}(X, X') \) the space of all invertible \( k \)-jets sending \( x \) into \( x' \). The subset \( G^k_x(X) \subset J^k_{x,x}(X, X) \) of all invertible \( k \)-jets forms an algebraic group with respect to composition. Completely analogously \( k \)-jets of smooth maps between smooth real manifolds \( M \) and \( M' \) are defined, for which we shall use the same notation \( J^k(M, M') \). The possible confusion will be eliminated by the convention that we write \( X_\mathbb{R} \) whenever we consider a complex manifold \( X \) as a real manifold. Thus, if \( X \) and \( X' \) are complex manifolds, \( J^k(X, X') \) is the space of all \( k \)-jets of holomorphic maps and \( J^k(X_\mathbb{R}, X'_\mathbb{R}) \) is the space of all \( k \)-jets of smooth maps.
Furthermore, we shall need \( k \)-jets of real submanifolds of fixed dimension of a smooth real manifold \( M \). Let \( \mathcal{C}_{k}^{m}(M) \) be the set of all germs at \( x \) of real \( C^{k} \)-smooth \( m \)-dimensional submanifolds of \( M \) through \( x \). We say that two germs \( V, V' \in \mathcal{C}_{k}^{m}(M) \) are \( k \)-equivalent, if, in a local coordinate neighborhood of \( 0 \), the union \( \bigcup_{x \in M} J_{x}^{k,m}(M) \) with the natural fiber bundle structure over \( M \). Furthermore, for any real \( C^{k} \)-smooth \( m \)-dimensional submanifold \( V \subset M \) through \( x \), denote by \( J^{k,m}(V) \in J_{x}^{k,m}(M) \) the corresponding \( k \)-jet. The space \( J_{x}^{k,m}(M) \) carries a natural real (nonsingular) algebraic variety structure.

We now introduce the notions of equivalence and rigidity that will be crucial in the sequel.

**Definition 2.1.**

1. Two \( k \)-jets of real submanifolds of the same dimension \( \Lambda_{j} \in J_{p_{j}}^{k,m}(\mathbb{C}_{R}^{n}), j = 1, 2 \), are called biholomorphically equivalent if there exists a germ of a biholomorphic map \((\mathbb{C}^{n}, p_{j}) \to (\mathbb{C}^{n}, p_{2}) \) sending \( \Lambda_{1} \) to \( \Lambda_{2} \).

2. A \( C^{k} \)-smooth generic submanifold \( M \subset \mathbb{C}^{n} \) is called totally rigid of order \( k \), if for any \( p_{1} \neq p_{2} \in M \), the jets \( J_{p_{1}}^{k}(M) \) and \( J_{p_{2}}^{k}(M) \) are not biholomorphically equivalent in the sense of (1).

3. A \( k \)-jet \( \Lambda \in J_{p}^{k,m}(\mathbb{C}_{R}^{n}) \) is called totally rigid if any \( C^{k} \)-smooth submanifold passing through \( p \) and having \( \Lambda \) as its \( k \)-jet at \( p \), contains a neighborhood of \( p \) that is totally rigid of order \( k - 1 \) in the sense of (2).

**Example 2.2.** Any 0-jets of real submanifolds at \( p \) are obviously biholomorphically equivalent and 1-jets are equivalent if and only if their CR-dimensions at \( p \) are the same. Two 2-jets of generic submanifolds are equivalent if and only their Levi forms at \( p \) are linearly equivalent (e.g. of the same rank and signature in the hypersurface case). Furthermore it follows from the Chern-Moser theory [CM74] that two \( k \)-jets of Levi-nondegnerate hypersurfaces of the same signature are always biholomorphically equivalent for \( k \leq 5 \) in case \( n = 2 \) and for \( k \leq 3 \) in case \( n > 2 \), but may not be equivalent in general for \( k \) larger.

It is crucial for our method to consider the total rigidity of order \( k - 1 \) in (3) (rather then e.g. of order \( k \)) for the representing submanifolds \( M \) with \( J_{p}^{k}(M) = \Lambda \). This allows us to achieve the total rigidity of \( M \) of order \( k - 1 \) near \( p \) by ensuring that the first order derivatives of \( J_{q}^{k-1}(M) \) at \( p \) as function of \( q \in M \) have suitable transversality property with respect to the submanifolds (orbits) of biholomorphically equivalent \((k-1)\)-jets (see the proof of Proposition 2.3 below for more details). Thus we need \( \Lambda \) to be of higher order than \( k - 1 \) to include the extra derivatives. More precisely, the existence of totally rigid jets is guaranteed by the following statement.

**Proposition 2.3.** For fixed integers \( n < m < 2n \), a point \( p \in \mathbb{C}^{n} \) and sufficiently large \( k \) (depending on \( n \) and \( m \) but not on \( p \)), the set of all totally rigid \( k \)-jets in \( J_{p}^{k,m}(\mathbb{C}_{R}^{n}) \) contains an open dense subset.
In fact we show that the number $k$ in Proposition 2.3 can be chosen such that the following inequality holds:

$$
(2n - m) \left( \binom{k + m - 1}{m} - 1 \right) - 2n \left( \binom{k + n - 1}{n} - 1 \right) \geq m. \tag{2.1}
$$

The proof will be based on the following lemmas (of which the first is standard and provided with a short proof for the reader’s convenience). We write $\| \cdot \|_{C^l}$ for the standard $C^l$ norm.

**Lemma 2.4.** Let $\varphi: \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map and $V$ be an open neighborhood in $\mathbb{R}^n$ of a point $a \in \mathbb{R}^n$. Then for any $\epsilon > 0$ and any integers $0 \leq k \leq l$, there exists $\delta > 0$ such that, if $\Lambda \in J^k_a(\mathbb{R}^n, \mathbb{R}^m)$ is a $k$-jet with $\|\Lambda - j^k_a\varphi\| < \delta$, then there exists another smooth map $\tilde{\varphi}: \mathbb{R}^n \to \mathbb{R}^m$ such that $j^k_a\tilde{\varphi} = \Lambda$, $\tilde{\varphi}|_{\mathbb{R}^n \setminus V} = \varphi|_{\mathbb{R}^n \setminus V}$ and $\|\tilde{\varphi} - \varphi\|_{C^l} < \epsilon$.

**Proof.** Without loss of generality, $V$ is bounded. We shall look for a map $\tilde{\varphi}: \mathbb{R}^n \to \mathbb{R}^m$ of the form

$$
\tilde{\varphi}(x) := \varphi(x) + \chi(x) \cdot (\psi(x) - \varphi(x)), \tag{2.2}
$$

where $\psi: \mathbb{R}^n \to \mathbb{R}^m$ is a smooth map with $j^k_\psi = \Lambda$ and $\chi: \mathbb{R}^n \to \mathbb{R}$ is a fixed smooth function which is 1 in a neighborhood of $a$ and 0 outside $V$. Then $j^k_\tilde{\varphi} = \Lambda$ and $\tilde{\varphi}|_{\mathbb{R}^n \setminus V} = \varphi|_{\mathbb{R}^n \setminus V}$. Furthermore, there exists $C > 0$ depending on $\chi$ but not on $\psi$ such that

$$
\|\tilde{\varphi} - \varphi\|_{C^l} < C\|\psi - \varphi\|_{C^l}.
$$

If $\delta$ is sufficiently small and $\Lambda$ satisfies our assumption, we can always choose $\psi$ with $\|\psi - \varphi\|_{C^l} < \epsilon/C$ on the closure $\overline{V}$. Then the map $\tilde{\varphi}$ given by (2.2) satisfies the required properties. \qed

**Lemma 2.5.** Let $k, l, s \geq 0$ and $0 \leq r \leq m$ be any integers, $U \subset J^l(\mathbb{R}^n, \mathbb{R}^s)$ be an open set and $F: U \to \mathbb{R}^r$ be a smooth map of constant rank $r$. Then, for any nonempty open set $B \subset \mathbb{R}^m$ and a smooth map $f: B \to \mathbb{R}^s$ satisfying $j^l_xf \in U$ for all $x \in B$, there exists another nonempty open subset $\tilde{B} \subset B$ and another smooth map $\hat{f}: \tilde{B} \to \mathbb{R}^s$ such that the following holds:

1. the $C^k$ norm of $\hat{f} - (f|_{\tilde{B}})$ can be chosen arbitrarily small;
2. $j^l_x\hat{f} \in U$ for all $x \in \tilde{B}$;
3. the map $x \in \tilde{B} \mapsto F(j^l_x\hat{f}) \in \mathbb{R}^r$ is also of constant rank $r$.

**Proof.** Without loss of generality, the integer $k$ (used only in (1)) is $\geq l$. We prove the lemma by induction on $r$. For $r = 0$ the statement is trivial. Suppose it holds for any $r < r_0 \leq m$ and we are given a map $F: U \to \mathbb{R}^{r_0}$ as in the lemma. Consider the standard splitting $\mathbb{R}^{r_0} = \mathbb{R}^{r_0-1} \times \mathbb{R}$ and the corresponding components $F_1: U \to \mathbb{R}^{r_0-1}$ and $F_2: U \to \mathbb{R}$ of $F$ (so that $F = (F_1, F_2)$). Then the induction assumption for $F_1$ yields a map $\hat{f}: \tilde{B} \to \mathbb{R}^{r_0-1}$ such that $\|\hat{f} - (f|_{\tilde{B}})\|_{C^k}$ is arbitrarily small, $j^l_x\hat{f} \in U$ for all $x \in \tilde{B}$ and the map $\Phi_1(x) := F_1(j^l_x\hat{f}) \in \mathbb{R}^{r_0-1}$ is of constant rank $r_0 - 1 < m$ for $x \in \tilde{B}$. By shrinking $\tilde{B}$ if
necessary, we may assume it is connected and of the form \( \widehat{B} = \widehat{B}_1 \times \widehat{B}_2 \subset \mathbb{R}^{m-r_0+1} \times \mathbb{R}^{r_0-1} \) and such that, for some \( c \in \mathbb{R}^{r_0-1} \), the level set \( \mathcal{C} := \{ x \in \widehat{B} : \Phi_1(x) = c \} \) is a graph of a smooth map \( \varphi : \widehat{B}_1 \to \widehat{B}_2 \).

We now consider two points \( x_1 \neq x_2 \in \mathcal{C} \) and look for a small perturbation \( \tilde{f} \) of \( f \) and \( \tilde{x}_2 \) of \( x_2 \) such that \( x_1 \) and \( \tilde{x}_2 \) still belong to the same level set \( \tilde{C} = \{ x \in \widehat{B} : \Phi_1(x) = \Phi_1(\tilde{x}_2) \} \). More precisely, suppose that \( F(j^{1}_{x_1} \tilde{f}) = F(j^{1}_{x_2} \tilde{f}) \) (otherwise no perturbation is needed). By the assumption of the lemma, the map \( F = (F_1, F_2) \) has constant rank \( r_0 \) in \( U \). Since \( r_0 \leq m \leq \dim U \), we can find a point \( \tilde{x}_2 \in \widehat{B} \) and a jet \( \Lambda \in U \cap J^1_{\tilde{x}_2}(\mathbb{R}^m, \mathbb{R}^n) \) arbitrarily close to \( j^{1}_{\tilde{x}_2} \tilde{f} \) such that \( F_1(\Lambda) = F_1(j^{1}_{\tilde{x}_2} \tilde{f}) \) but \( F_2(\Lambda) \neq F_2(j^{1}_{x_2} \tilde{f}) \). Since \( \Lambda \) can be chosen arbitrarily close to \( j^{1}_{\tilde{x}_2} \tilde{f} \), it can be represented by a smooth map \( \tilde{f} \) that is arbitrarily close to \( \tilde{f} \) in the \( C^k \)-norm and differs from it on a neighborhood of \( x_2 \) with compact support in \( \widehat{B} \setminus \{ x_1 \} \) in view of Lemma 2.4. By choosing the norm \( \| f - \tilde{f} \|_{C^k} \) sufficiently small, we shall preserve the properties that \( j^{1}_{x_1} \tilde{f} \in U \) for all \( x \in \widehat{B} \), the rank of \( \Phi_1(x) := F_1(j^{1}_{x_1} \tilde{f}) \) is still \( r_0 - 1 \) and the level set \( \mathcal{C} := \{ x \in \widehat{B} : \Phi_1(x) = c \} \) is still a graph of a smooth map \( \varphi : \widehat{B}_1 \to \widehat{B}_2 \). Hence \( \mathcal{C} \) is a connected manifold containing two points \( x_1, \tilde{x}_2 \in \mathcal{C} \) such that \( F_2(j^{1}_{x_2} \tilde{f}) \neq F_2(j^{1}_{\tilde{x}_2} \tilde{f}) \). The latter fact implies that the function \( \Phi_2(x) := F_2(j^{1}_{x} \tilde{f}) \) is not constant on \( \mathcal{C} \) and therefore its differential is somewhere nonzero. Putting this property together with the rank property of \( \Phi_1 \), we conclude that the rank of \( \Phi(x) := F(j^{1}_{x} \tilde{f}) \) is \( r_0 \) at some point \( x_0 \in \widehat{B} \). The required conclusion is obtained by replacing \( \tilde{f} \) with \( \tilde{f} \) and \( \widehat{B} \) with a sufficiently small open neighborhood of \( x_0 \).

Proof of Proposition 2.3. We may assume \( p = 0 \). Consider the natural action of the group \( \mathbb{G}_0^{-1}(\mathbb{C}^n) \) (consisting of all \( (k-1) \)-jets at 0 of local biholomorphic maps of \( \mathbb{C}^n \)) on the space \( J^{1-m}_{0}(\mathbb{C}^n) \) (consisting of all \( (k-1) \)-jets at 0 of real \( m \)-dimensional submanifolds of \( \mathbb{C}^n \) passing through 0). The dimensions of the jet spaces can be computed directly:

\[
\dim_{\mathbb{R}} \mathbb{G}_0^{-1}(\mathbb{C}^n) = 2n \left( \binom{k+n-1}{n} - 1 \right), \quad \dim_{\mathbb{R}} J^{1-m}_{0}(\mathbb{C}^n) = (2n-m) \left( \binom{k+m-1}{m} - 1 \right).
\]

Hence the inequality (2.1) is equivalent to

\[
\dim_{\mathbb{R}} J^{1-m}_{0}(\mathbb{C}^n) - \dim_{\mathbb{R}} \mathbb{G}_0^{-1}(\mathbb{C}^n) \geq m.
\]

In particular, for any sufficiently large \( k \), all orbits of \( \mathbb{G}_0^{-1}(\mathbb{C}^n) \) in \( J^{1-m}_{0}(\mathbb{C}^n) \) have their (real) codimension at least \( m \). In the rest of the proof we shall assume that (2.4) is satisfied.

It is easy to see that \( \mathbb{G}_0^{-1}(\mathbb{C}^n) \) is an algebraic group acting rationally on \( J^{1-m}_{0}(\mathbb{C}^n) \) by calculating the group operation and the action in local coordinates. Hence the orbits of
$G_0^{k-1}(\mathbb{C}^n)$ form, on an open dense subset $\Omega \subset J_0^{k-1,m}(\mathbb{C}_R^n)$, a foliation into real submanifolds of a fixed constant codimension $\geq m$. Consider any $k$-jet $\Lambda_0 \in J_0^{k,m}(\mathbb{C}_R^n)$ represented by the graph of a $C^\infty$-smooth map $\varphi_0: \mathbb{R}^m \to \mathbb{R}^{2n-m}$ with $\varphi(0) = 0$, where we choose a suitable identification of $\mathbb{C}_R^n$ with $\mathbb{R}^m \times \mathbb{R}^{2n-m}$ (after a possible permutation of the real coordinates). Thus $\Lambda_0 = j_0^k((id \times \varphi_0)(\mathbb{R}^m))$. By the density of $\Omega$, we can find another $C^\infty$-smooth map $\varphi: \mathbb{R}^m \to \mathbb{R}^{2n-m}$ with $\varphi(0) = 0$ and $\Theta := j_0^k\varphi$ arbitrarily close to $j_0^k\varphi_0$ such that the $(k-1)$-jet $\Lambda \in J_0^{k-1,m}(\mathbb{C}_R^n)$ at 0 of the graph of $\varphi$ is contained in $\Omega$. (By this choice, also $\Lambda$ is arbitrarily close to $\Lambda_0$.) We can now find an open neighborhood $U$ of $j_0^k\varphi$ in $J^{k-1}(\mathbb{R}^m, \mathbb{R}^{2n-m})$ and a smooth map $F: U \to \mathbb{R}^m$ of constant rank $m$ and constant on the orbits (recall that $m$ does not exceed the orbit codimension), such that two $(k-1)$-jets $\Lambda_j \in J_j^{k-1,m}(\mathbb{C}_R^n)$, $j = 1, 2$, near $\Lambda_0$ (where $(x_j, x'_j) \in \mathbb{R}^m \times \mathbb{R}^{2n-m}$), which are represented by graphs of some smooth maps $\varphi_1, \varphi_2: \mathbb{R}^m \to \mathbb{R}^{2n-m}$, are biholomorphically inequivalent (in the sense of Definition 2.1) whenever $F(j^k_{x_1}\varphi_1) \neq F(j^k_{x_2}\varphi_2)$. Here $F$ can be obtained by taking the first $m$ coordinates in any real coordinate system $(x, y) \in \mathbb{R}^m \times \mathbb{R}^2$, for which the orbits are given by $x = \text{const}$. Then Lemma 2.5 can be applied to $U$, $F$, $f := \varphi$ and an arbitrarily small neighborhood of $B \subset \mathbb{R}^m$ of 0. Let $\tilde{B}$ and $\tilde{f}: \tilde{B} \to \mathbb{R}^{2n-m}$ be given by the lemma.

We claim that, for any $x_0 \in \tilde{B}$, the $k$-jet $\Lambda(x_0) \in J^{k,m}(\mathbb{C}_R^n)$ of the graph of $\tilde{f}$ at $(x_0, \tilde{f}(x_0))$ is totally rigid in the sense of Definition 2.1 (3). Indeed, fix any $x_0 \in \tilde{B}$ and consider any $C^k$-smooth real $m$-dimensional submanifold $V \subset \mathbb{C}_R^n$ passing through $(x_0, \tilde{f}(x_0))$ with $j^k_{x_0}\tilde{f}(x_0))(V) = \Lambda(x_0)$. By shrinking $V$, if necessary, we may assume that $V$ is a graph of a smooth map $g: B(x_0) \to \mathbb{R}^{2n-m}$, where $B(x_0)$ is a suitable open neighborhood of $x_0$. Then $j^k_{x_0}g = j^k_{x_0}\tilde{f}$ and therefore, the ranks of the maps $x \mapsto F(j^k_xg)$ and $x \mapsto F(j^k_x\tilde{f})$ coincide at $x = x_0$. (Here is the step, where we use the different integers $k$ and $k-1$ for $x_0$ and points nearby respectively.) By property (3) in Lemma 2.5, the rank of the second map is $m$ and hence, so is the rank of the first map. But the latter fact implies that $F(j^k_{x_1}g) \neq F(j^k_{x_2}g)$ for any $x_1 \neq x_2$ sufficiently close to $x_0$. In view of the choice of $F$, it follows that the jets $j^k_{x_1,g(x_1)}(V)$ and $j^k_{x_2,g(x_2)}(V)$ are biholomorphically inequivalent for any such $x_1 \neq x_2$, which is precisely what is needed to show that $\Lambda(x_0)$ is totally rigid. It remains to observe, that any translation of $\Lambda(x_0)$ is also totally rigid, hence we can find totally rigid $k$-jets also in $J_0^{k,m}(\mathbb{C}_R^n)$ arbitrarily close to the original jet $\Lambda_0$. The proof is complete. □

3. Realization of certain groups as CR stability groups

In the sequel we shall use the same letter for a germ and its representative unless there will be a danger of confusion. We begin with a standard lemma, whose proof is given here for the reader’s convenience.
Lemma 3.1. Let \((G_k)_{k \geq 1}\) be an increasing sequence of finite groups of germs of local biholomorphic maps of \(\mathbb{C}^n\) in a neighborhood of a point \(p\) fixing that point. Let \(M\) be a smooth real submanifold of \(\mathbb{C}^n\) passing through \(p\). Let \((D_k)_{k \geq 1}\) be a sequence of domains containing \(p\) and such that, for each \(k\), the germs from \(G_k\) can be represented by biholomorphic self-maps of \(D_k\). Then there exist a sequence of points \(p_k \in M, k \geq 1\), converging to \(p\) and a sequence of mutually disjoint open neighborhoods \(V_k\) of \(p_k\) in \(\mathbb{C}^n\) such that \(\overline{V}_k \subset D_k\) and, if \(g(\overline{V}_k) \cap \overline{V}_l \neq \emptyset\) for some \(k, l\) and \(g \in G_k\), then necessarily \(k = l\) and \(g \equiv \text{id}\).

Note that the existence of domains \(D_k\) easily follows from the finiteness of each \(G_k\). Indeed, if \(\tilde{D}_k\) is any domain where all germs from \(G_k\) biholomorphically extend, it suffices to take \(D_k := \bigcap_{g \in G_k} g(\tilde{D}_k)\).

Proof. We shall construct \(p_k \in M\) and \(V_k \subset \mathbb{C}^n\) inductively. Let \(k = 1\). Since \(G_1\) is finite, the set of points \(x \in D_1\), such that there are two elements \(g_1 \neq g_2 \in G_1\) with \(g_1(x) = g_2(x)\), is a complement of a proper analytic subset. Hence we can choose \(p_1 \in M\) and a neighborhood \(V_1 \subset D_1\) of \(p_1\) in \(\mathbb{C}^n\) with \(p \not\in \overline{V}_1\) such that \(g(\overline{V}_1) \cap \overline{V}_1 \neq \emptyset\) for \(g \in G_1\)

Now suppose that \(p_l\) and \(V_l\) with \(p \not\in \overline{V}_l\) have been chosen for all \(l < k\). Since \(G_k\) is finite, we can choose a neighborhood \(U\) of \(p\) in \(D_k\) such that \(g(U) \cap \overline{V}_l = \emptyset\) for all \(l < k\) and \(g \in G_k\). Using the same argument as before, we can choose \(p_k\) arbitrarily close to \(p\) and \(V_k\) with \(p \not\in \overline{V}_k\) such that \(p_k \in V_k \subset \overline{V}_k \subset U\) and, for any \(g \neq \text{id} \in G_k\), \(g(\overline{V}_k) \cap \overline{V}_k = \emptyset\).

Since \(\overline{V}_k \subset U\) and \(g(U) \cap \overline{V}_l = \emptyset\) for all \(l \leq k\) and \(g \in G_k\), it follows that \(g(\overline{V}_k) \cap \overline{V}_l \neq \emptyset\) can hold for some \(l \leq k\) and \(g \in G_k\) if and only if \(k = l\) and \(g \equiv \text{id}\). It is easy to see that the sequences \((p_k)\) and \((V_k)\) so constructed satisfy the required properties. \(\square\)

Proof of Theorem 1.1. Since the statement is local, we fix an identification \(\mathbb{C}^n \cong \mathbb{R}^m \times \mathbb{R}^{2n-m}\) near \(p\) such that \(M\) is represented by the graph of a smooth map \(\varphi : \mathbb{R}^m \to \mathbb{R}^{2n-m}\) with \(\|\varphi\|_{C^1}\) sufficiently small. We shall write \(B_r(a)\) for the open ball with center \(a\) and radius \(r\) with respect to the product metric of the Euclidean metrics of \(\mathbb{R}^m\) and \(\mathbb{R}^{2n-m}\). With this choice of the metric on \(\mathbb{R}^m \times \mathbb{R}^{2n-m}\) and \(\|\varphi\|_{C^1}\) sufficiently small, we have the property that, for any \(a \in M\), the intersection \(B_r(a) \cap M\) coincides with the graph of \(\varphi\) over the projection of \(B_r(a)\) to \(\mathbb{R}^m\) (which is an Euclidean ball in \(\mathbb{R}^m\)). In the course of the proof we shall consider small perturbations of \(M\) obtained as graphs of small perturbations of \(\varphi\). We shall always assume that the \(C^1\) norms of these perturbations are still small, so that the mentioned relation between ball intersections with their graphs and graphs over balls still holds.

By the assumption, \(G = \bigcup_k G_k\), where \(G_k, k \geq 1\), is an increasing sequence of finite groups of local biholomorphic maps of \(\mathbb{C}^n\) in a neighborhood of \(p\), fixing \(p\) and preserving the germ \((M, p)\). As indicated above, we can choose a decreasing sequence of open neighborhoods \(D_k\) of \(p\) in the unit ball \(B_1(p)\) in \(\mathbb{C}^n\) centered at \(p\), such that, for each \(k\), all
of Definition 2.1). Then it follows from (2.3) that, for each $k$, $D_k$, $V_k$ inductively such that, in addition,

$$\max \left( \sup_{z \in D_{k+1}, g \in G_k} \|g'(z)\|, \sup_{z \in D_{k}, g \in G_k} \|g'(z)\| \right) \frac{\sup_{z \in V_{k+1}} |z - p|}{\inf_{z \in V_k} |z - p|} \to 0, \quad k \to \infty, \quad (3.1)$$

where $g'$ is the Jacobian matrix of $g$.

For an $l$-jet $\Lambda \in J_{x}^{l,m}(\mathbb{C}_{\mathbb{R}}^n)$, we denote by $\text{Orb}(\Lambda) \subset J_{y}^{l,m}(\mathbb{C}_{\mathbb{R}}^n)$ the set of all $l$-jets $\Lambda \in J_{y}^{l,m}(\mathbb{C}_{\mathbb{R}}^n)$ for all $y \in \mathbb{C}_{\mathbb{R}}^n$ that are biholomorphically equivalent to $\Lambda$ (in the sense of Definition 2.1). Then it follows from (2.3) that, for $l$ sufficiently large, the subset $\bigcup_{z \in M \cap D_1} \text{Orb}(j^l_x M) \subset J_{y}^{l,m}(\mathbb{C}_{\mathbb{R}}^n)$ has Lebesgue measure zero and therefore its complement is dense in $J_{y}^{l,m}(\mathbb{C}_{\mathbb{R}}^n)$. The same argument obviously applies to any other real submanifold of $\mathbb{C}^n$ of the same dimension as $M$. We shall use this property to choose jets that are not in the certain unions of orbits. We shall consider $l$ sufficiently large so that this choice is always possible.

We next consider a sequence $\varepsilon_k$, $0 < \varepsilon_k < 1$, converging to 0 and such that

$$\varepsilon_k \left( \sup_{x \in D_k, g \in G_k} \|g'(x)\| \right) \to 0, \quad k \to \infty. \quad (3.2)$$

Then, as a consequence of Lemma 2.4 and Proposition 2.3, we can find a sequence of neighborhoods $U_k$ of $p_k$ in $V_k$ with $\overline{U_k} \subset V_k$ and a sequence of graphs $N_k$ of smooth maps $\varphi_k: \mathbb{R}^m \to \mathbb{R}^{2n-m}$ with

$$\|\varphi_k - \varphi\|_{C^k} < \varepsilon_k, \quad N_k \setminus \overline{U_k} = M \setminus \overline{U_k}, \quad (3.3)$$

and such that the following holds. There exist points $q_k \in N_k$ and real numbers $\delta_k > 0$ such that, for each $k$, the $2\delta_k$-neighborhood of $q_k$ in $N_k$, $B_{2\delta_k}(q_k) \cap N_k$, is contained in $U_k$, totally rigid (in the sense of Definition 2.1) and

$$j^l_x N_k \notin \bigcup_{x \in M \cap D_1} \text{Orb}(j^l_x M). \quad (3.4)$$

As the next step, we define $X_k \subset N_k \cap U_k$ to be the subset of all points $y \neq q_k$ for which there exists a CR-diffeomorphism from $B_{2\delta_k}(y) \cap N_k$ into $B_{2\delta_k}(q_k) \cap N_k$, sending $y$ into $q_k$. Since the CR-manifold $B_{2\delta_k}(q_k) \cap N_k$ is totally rigid by our construction, it is clear that for any $y_1 \neq y_2 \in X_k$, the neighborhood $B_{2\delta_k}(y_1) \cap N_k$ cannot contain $y_2$. Hence the neighborhoods $B_{2\delta_k}(y) \cap N_k$, $y \in X_k$, do not intersect and so $X_k$ must be a finite set. It is also clear that $X_k \cap B_{2\delta_k}(q_k) \cap N_k = \emptyset$ again by the total rigidity of $B_{2\delta_k}(q_k) \cap N_k$. Furthermore we must have $X_k \subset N_k \cap U_k$ in view of (3.3) and (3.4).

We next choose a sequence $\eta_k$, $0 < \eta_k < \varepsilon_k \delta_k$ and apply again Lemma 2.4 to obtain a sequence of graphs $M_k$ of smooth maps $\psi_k: \mathbb{R}^m \to \mathbb{R}^{2n-m}$ with

$$\|\psi_k - \varphi_k\|_{C^k} < \varepsilon_k \quad (3.5)$$
and finite subsets $\tilde{X}_k \subset M_k$ with
\[ j'_y M_k \notin \bigcup_{x \in N_k \cap D_1} \text{Orb}(j'_x N_k) \quad \forall y \in \tilde{X}_k, \tag{3.6} \]
and such that, for $W_k := \bigcup_{g \in \tilde{X}_k} B_{\eta_k}(y),$
\[ X_k \cap M_k = \emptyset, \quad M_k \setminus W_k = N_k \setminus \overline{W_k}, \quad \overline{W_k} \cap B_{\delta_k}(g_k) = \emptyset. \tag{3.7} \]
We may in addition assume that $\eta_k$ is sufficiently small so that $\overline{W_k} \subset V_k.$

Finally, we define the new generic submanifold $\tilde{M} \subset \mathbb{C}^n$ by replacing $g(M \cap V_k)$ with $g(M_k \cap V_k)$ for every sufficiently large $k$ and every $g \in G_k,$ i.e.
\[ \tilde{M} := \left( M \setminus \bigcup_{k \geq k_0, g \in G_k} g(M \cap V_k) \right) \bigcup \left( \bigcup_{k \geq k_0, g \in G_k} g(M_k \cap V_k) \right), \tag{3.8} \]
where $k_0$ is sufficiently large. (Note that all neighborhoods $g(V_k), g \in G_k,$ $k \geq k_0,$ are disjoint together with their closures since they are given by Lemma 3.1.) Then $\tilde{M}$ is a smooth submanifold through $p$ and, if $\varepsilon_k$ have been chosen sufficiently rapidly converging to 0, $\tilde{M}$ is tangent to $M$ of infinite order at $p$ in view of (3.3) and (3.5). Consequently $\tilde{M}$ is also of finite type at $p.$ Furthermore, the germ $(\tilde{M}, p)$ is clearly invariant under the action of $G,$ i.e. the group $\text{Aut}_{\text{CR}}(\tilde{M}, p)$ of germs at $p$ of all local CR-automorphisms of $M$ fixing $p$ contains $G.$

We now claim that $\text{Aut}_{\text{CR}}(\tilde{M}, p) = G.$ Indeed, fix any $f \in \text{Aut}_{\text{CR}}(\tilde{M}, p)$ and its representative defined in some neighborhood of $p$ in $\tilde{M},$ denoted by the same letter. Then for $k$ sufficiently large, $f$ is defined in $D_k \subset \tilde{M}$ with $f(D_k \cap \tilde{M}) \subset D_1.$ By the construction, each $g_k \in N_k \cap U_k$ is not contained in $W_k,$ hence it is in $M_k \cap V_k$ and therefore in $\tilde{M}$ so that we can evaluate $f(g_k).$ Then (3.4) implies that $f(g_k) \notin \tilde{M}$ and hence $f(g_k) \in g(U_s) \subset g(V_s)$ for some $s$ and some $g \in G_s.$ Thus we have the estimates
\[ (\sup_{D_s} \|(g^{-1})'\|)^{-1} \frac{\inf_{z \in V_s} |z - p|}{\sup_{z \in V_s} |z - p|} \leq \frac{|f(g_k) - p|}{|q_k - p|} \leq \sup_{D_s} \|g'\| \frac{\inf_{z \in V_k} |z - p|}{\sup_{z \in V_k} |z - p|}. \tag{3.9} \]
On the other hand, since $f$ is a local diffeomorphism of $\tilde{M}$ fixing $p,$ there exist constants $0 < c < C$ such that
\[ c \leq \frac{|f(z) - p|}{|z - p|} \leq C, \quad c \leq \|f'(z)\| \leq C \tag{3.10} \]
for $z \neq p \in \tilde{M}$ sufficiently close to $p.$ Then if $k$ is sufficiently large, we must have $s = k$ in view of (3.1). Hence $f(g_k) \in g_k(U_k)$ for suitable $g_k \in G_k.$ Setting $f_k := g_k^{-1} \circ f \in \text{Aut}_{\text{CR}}(M, p),$ we have $f_k(g_k) \in U_k \cap \tilde{M}.$ In view of (3.8), this means $f_k(g_k) \in U_k \cap M_k.$

We now claim that, for $k$ sufficiently large, we must have $B_{\varepsilon_k \delta_k}(f_k(g_k)) \cap \tilde{M} \subset N_k.$ Indeed, otherwise, in view of (3.7), we would have $W_k \cap B_{\varepsilon_k \delta_k}(f_k(g_k)) \neq \emptyset$ and, since $\eta_k < \varepsilon_k \delta_k,$ it would imply $\tilde{X}_k \cap B_{2\varepsilon_k \delta_k}(f_k(g_k)) \neq \emptyset.$ However, in view of (3.2) and (3.10),
this would mean that, for \( k \) sufficiently large and some point \( y \in \tilde{X}_k \), the inclusion \( f_k^{-1}(y) \subset B_{\delta_k}(q_k) \cap \tilde{M} \) would hold. By our construction, \( B_{\delta_k}(q_k) \cap \tilde{M} \subset N_k \) and hence we would have a contradiction with (3.6). Hence we have \( B_{\varepsilon_k \delta_k}(f_k(q_k)) \cap \tilde{M} \subset N_k \) as claimed. Thus \( B_{\varepsilon_k \delta_k}(f_k(q_k)) \cap \tilde{M} = B_{\varepsilon_k \delta_k}(f_k(q_k)) \cap N_k \). Again, using (3.2) and (3.10), we conclude that, for \( k \) sufficiently large, \( f_k^{-1} \) sends \( B_{\varepsilon_k \delta_k}(f_k(q_k)) \cap N_k \) into \( B_{\delta_k}(q_k) \cap N_k \). By our construction of the set \( X_k \subset N_k \), the latter conclusion means either \( f_k(q_k) \in X_k \) or \( f_k(q_k) = q_k \). The first case is impossible in view of the first condition in (3.7). Hence we have \( f_k(q_k) = q_k \) and, since a neighborhood of \( q_k \) in \( M_k \) is totally rigid, this means \( f_k \equiv \text{id} \) in a neighborhood of \( q_k \). Since \((M, p)\) is of finite type, we have \( f_k \equiv \text{id} \) as germs at \( p \), and hence \( f \equiv g_k \in G \) implying the desired conclusion. \( \square \)

**References**


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