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DECOMPOSITION OF CR-MANIFOLDS AND SPLITTING OF CR-MAPS

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Abstract. We show the uniqueness of local and global decompositions of abstract CR-manifolds into direct products of irreducible factors and splitting property for their CR-diffeomorphisms into direct products with respect to these decompositions. The assumptions on the manifolds are finite nondegeneracy and finite type on a dense subset. In the real-analytic case, these are the standard assumptions that appear in many other questions. In the smooth case, the assumptions cannot be weakened by replacing "dense" with "open" as is demonstrated by an example. An application to the cancellation problem is also given. The proof is based on the development of methods of [BER99b, BRZ00, KZ01] and the use of "approximate infinitesimal automorphisms" introduced in this paper.

Contents

1. Introduction

Decompositions of various types of manifolds into direct products of submanifolds play an important role in their study. For instance, for semisimple Lie groups and for symmetric spaces, such decompositions are crucial for the classification. In Riemannian geometry such a decomposition is known as de Rham decomposition (see [KN96]). In all these cases the corresponding decomposition is unique unless there are present so-called "flat factors" whose classification is simple. Most geometric and functional-theoretic questions for the manifolds then are reduced to the irreducible factors.

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In this paper we study local decompositions of germs of (abstract) CR-manifolds into irreducible factors as well as their global analogues and establish their uniqueness. Here the role of "flat factors" is played by Levi-flat directions, where, in general, "higher order Levi forms" have to be taken into account. The simplest example is given by the Levi-flat manifold $M = \mathbb{R} \times \mathbb{C}$ where the choice of the factor $\mathbb{R}$ is obviously not unique. There are different known nondegeneracy conditions to exclude such phenomenon, most of them are usually formulated for real-analytic CR-manifolds. Those that seem to be the easiest to transfer to the smooth case and also the easiest to compute are the condition of *finite nondegeneracy* (see [H83, BHR96, E98]) and of *finite type* (in the sense of Kohn [K72] and Bloom-Graham [BG77]). We refer to §2 for main definitions and mention here only that finite nondegeneracy and finite type are implied respectively by the nondegeneracy of the Levi form and by the condition that the span of all Levi form values is of maximal possible dimension.

Our main result states that, in order to have the unique decomposition property, it is sufficient to require finite nondegeneracy and finite type only on a dense subset. In this paper "smooth" will always mean C^∞. A germ of a smooth CR-manifold is called *irreducible*, if it is not CR-diffeomorphic to a direct product of two germs of smooth CR-manifolds of positive dimension. We prove:

Theorem 1.1. *Let (M,p) be a germ of a smooth CR-manifold which is finitely nondegenerate and of finite type on a dense subset. Then, up to permutations, there exists a unique decomposition*

$$(M,p) \cong (M_1,p_1) \times \cdots \times (M_m,p_m),$$

where each germ (M_j,p_j) is irreducible. Furthermore, if f is a (germ of a) smooth local CR-diffeomorphism between (M,p) and another (germ of a) smooth CR-manifold (M',p') and if $(M',p') \cong (M'_1,p'_1) \times \cdots \times (M'_{m'},p_{m'})$ is the corresponding decomposition into irreducible factors, then $m = m'$ and, after a permutation of the factors (M'_j,p'_j), f factors as a direct product of the form $f = f^1 \times \cdots \times f^m$, where $f^j \colon (M_j,p_j) \to (M'_j,p'_j)$ are (germs of) local CR-diffeomorphisms for $j = 1,\dots,m$.

If M is real-analytic, the assumption of finite nondegeneracy on a dense subset in Theorem 1.1 is equivalent to *holomorphic nondegeneracy* of its local analytic CR-embedding, i.e. to the nonexistence of holomorphic local one-parameter automorphism group (see [BER96, BER99a]). This assumption is optimal for points in general position in the following sense. If there is no dense subset where M is finitely nondegenerate, then the existence of a local one-parameter automorphism group implies that at a point of general position M is locally CR-isomorphic to a product of $\mathbb{C}$ and another CR-manifold. In this case it is easy to see that the decomposition in Theorem 1.1 at such a point is never unique.

If, on the other hand, there is no dense subset where M is of finite type, the situation is reduced to the finite type case by considering the CR-orbits (see [N66, S73, BER99a]).

In case M is real-analytic, the assumptions in Theorem 1.1 are clearly equivalent to M being finitely nondegenerate and of finite type at some sequence of points converging to p. If M is merely smooth, the second condition is essentially weaker and is not sufficient for the conclusion of Theorem 1.1 to hold as Example 2.1 below shows.

Our next result is the following global version of Theorem 1.1. Here we call a CR-manifold *(globally) irreducible* if it is not CR-diffeomorphic to a direct product of two smooth CR-manifolds of positive dimension.

Theorem 1.2. *Let M be a smooth CR-manifold which is finitely nondegenerate and of finite type on a dense subset. Then, up to permutations, there exists a unique decomposition*

$$M \cong M_1 \times \cdots \times M_m,$$

where each M_j is irreducible. Furthermore, if f is a smooth CR-diffeomorphism between M and another smooth CR-manifold M' and if and $M' \cong M'_1 \times \cdots \times M'_{m'}$ is the corresponding decomposition, then $m = m'$ and after a permutation of factors of M', f factors as a direct product of the form $f = f^1 \times \cdots \times f^m$, where $f^j \colon M_j \to M'_j$ are smooth CR-diffeomorphisms.

Again, also here Example 2.1 shows that in the assumptions cannot be weakened by replacing a dense subset by an open subset. As immediate applications of Theorems 1.1 and 1.2 we obtain the following cancellation result:

Corollary 1.3. *Let M_1, M_2 and S be CR-manifolds that are finitely nondegenerate and of finite type on their dense subsets. If $M_1 \times S$ and $M_2 \times S$ are CR-diffeomorphic, then M_1 and M_2 are also CR-diffeomorphic. Furthermore, if for some points $p_1 \in M_1$, $p_2 \in M_2$, $s \in S$, $(M_1, p_1) \times (S, s)$ and $(M_2, p_2) \times (S, s)$ are CR-diffeomorphic, then also (M_1, p_1) and (M_2, p_2) are CR-diffeomorphic.*

A key ingredient of the proofs of Theorems 1.1 and 1.2 consists in establishing a rigidity property for local CR-diffeomorphisms (Proposition 4.1) that roughly states that, under the assumptions of Theorem 1.1, any smooth family of local diffeomorphisms that is CR in all arguments, is necessarily constant. The proof of this fact is based on a realization of the space of infinitesimal CR-automorphisms as a totally real subspace in a suitable jet space. For (not infinitesimal) CR-automorphisms of real-analytic CR-manifolds fixing a reference point, such a realization has been obtained by Baouendi-Ebenfelt-Rothschild [BER99b, Theorem 4]. The method of [BER99b] is based on the local complexification of real-analytic CR-manifolds that may not exist for general abstract CR-manifolds as in our case. In [KZ01] the second and the third authors proposed a method of an approximate local complexification that has been used to obtain jet parametrizations of local CR-automorphisms and even of local automorphisms that are CR only up to some finite order. Using this method we can reduce the problem to the real-analytic case but, as in [KZ01], after such a reduction we have to consider not only infinitesimal CR-diffeomorphisms of the corresponding submanifold $\widetilde{M} \subset \mathbb{C}^N$ but also more general holomorphic vector fields that preserve $\widetilde{M}$ only up to finite order (at the reference point). We call these vector fields "approximate infinitesimal automorphisms". In contrast to usual infinitesimal automorphisms, the local flow of an approximate infinitesimal automorphism may not consist even of approximate automorphisms since they may not send the reference point into a point of $\widetilde{M}$. Thus we cannot reduce the problem to (not infinitesimal) automorphisms and instead adapt the technique of [BER99b, BRZ00, KZ01] directly to our case.

An outline of the paper is as follows. In §2 we review basic facts and definitions for CR-manifolds and give an example showing that the assumption of finite nondegeneracy and of finite type in Theorems 1.1 and 1.2 cannot be replaced by the same assumptions on a sequence of points. In §3 we define approximate infinitesimal automorphisms and establish their totally real realizations in jet spaces. In §4 we prove the rigidity property for local CR-automorphisms mentioned above. Finally, §5 and §6 are devoted to the proofs of Theorems 1.1 and 1.2. The arguments in these parts are partially inspired by [U81].

2. Preliminaries and an example

Recall that an *(abstract) smooth CR-manifold* is a smooth manifold M together with an involutive subbundle $T^{0,1}M$ of the complexified tangent bundle $TM \otimes \mathbb{C}$ such that $T^{0,1}M \cap T^{1,0}M = 0$, where $T^{1,0}M = \overline{T^{0,1}M}$. (Involutivity means here that Lie brackets of vector fields in $T^{0,1}M$ are again in $T^{0,1}M$.) Instead of prescribing $T^{0,1}M$ one can also consider a real subbundle T^cM of TM together with a complex structure J on T^c_pM for each $p \in M$ depending smoothly on p. The relation between $T^{0,1}M$ and (T^cM, J) is given by $T^{0,1}M = \{\xi + iJ\xi : \xi \in T^cM\}$. The reader is referred to the books [B91, BER99a] for basic properties of CR-manifolds.

A CR-manifold M is said to be of *finite type* at a point p (in the sense of Kohn [K72] and Bloom-Graham [BG77]) if all vector fields in $T^{0,1}M$ and $T^{1,0}M$ span together with their commutators the maximal possible space $T_pM \otimes \mathbb{C}$. The *type* ν of M at p is the minimal length of commutators needed to span the maximal space. In this case we say that M is of type ν at p or (M,p) is of type ν.

A CR-manifold M is called *finitely nondegenerate* at p (see [H83, BHR96, E98] and also [BER99a, §11.1]) if, for some integer $k \geq 1$,

$$\operatorname{span}_{\mathbb{C}}\{\mathcal{T}_{L_s}(\ldots \mathcal{T}_{L_2}(\mathcal{T}_{L_1}\theta)\ldots)(p) : 0 \leq s \leq k,\ L_j \in \Gamma(T^{0,1}M),\ \theta \in \Gamma(T^{*0}M)\} = T^{*1,0}_pM, \qquad (2.1)$$

where $T^{*0}M$ and $T^{*1,0}M$ denote the bundles of complex 1-forms that vanish on $T^cM \times \mathbb{C}$ and on $T^{0,1}M$ respectively and $\mathcal{T}_L$ is the Lie derivative along L. Recall that for any $(0,1)$ vector field L, the Lie derivative $\mathcal{T}_L$ leaves the space $\Gamma(T^{*1,0}M)$ invariant and is given there by $\mathcal{T}_L\omega = \mathsf{i}_L d\omega$, where i_L denotes the contraction. If the number k is minimal with the above property, we say that M is *k-nondegenerate* at p or (M,p) is *k-nondegenerate*.

The following example shows that the conclusion of Theorem 1.1 may not hold if (M,p) is only assumed to be finitely nondegenerate and of finite type at a sequence of points converging to p.

Example 2.1. Let $M_0 \subset \mathbb{C}^2_{z,w}$ be given by $\mathsf{Im}\, w = \lambda(\mathsf{Re}\, w)z\overline{z}$, where $\lambda(x)$ is a smooth function on $\mathbb{R}$ that is zero for $x \leq 0$ and positive for $x > 0$ and let $M_1 \subset \mathbb{C}^2_{z,w}$ be the quadric $\mathsf{Im}\, w = z\overline{z}$. Then $M := M_0 \times M_1$ is finitely nondegenerate (even Levi-nondegenerate) and of finite type at every point $(0, x, a, b) \in M$ with $x > 0$. However, the obvious decomposition of $(M, 0)$ as $(M_0, 0) \times (M_1, 0)$ is not unique. Indeed, let φ be a smooth real function on $\mathbb{R}$ that is one for $x \geq 0$ and greater than one for $x < 0$. Then the map

$$(z_0, w_0, z_1, w_1) \mapsto (z_0, w_0, \varphi(\mathsf{Re}\, w_0)z_1, (\varphi(\mathsf{Re}\, w_0))^2 w_1)$$

defines a CR-automorphism of M that does not preserve the given splitting $M = M_0 \times M_1$.

3. Approximate infinitesimal automorphisms

We begin by considering a germ (M, p) of a generic real-analytic submanifold in $\mathbb{C}^N$ with a vector-valued defining function $r = (r^1, \ldots, r^d)$. Recall that an *infinitesimal automorphism* of (M, p) in $\mathbb{C}^N$ is a germ at p of a holomorphic vector field L on $\mathbb{C}^N$ such that $\mathsf{Re}\, L$ is tangent to M, i.e. $\mathsf{Re}\, Lr = 0$ on M. More generally, we introduce the notion of an approximate infinitesimal automorphisms of a given order k. By definition, an *approximate infinitesimal automorphism of (M, p) of order k* is a germ at p of a holomorphic vector field L on $\mathbb{C}^N$ satisfying

$$\mathsf{Re}\, Lr(x) = o(|x-p|^k) \quad \text{as} \quad x \in M \to p.$$

We denoted by $\mathsf{aut}^k(M, p)$ the vector space of all approximate infinitesimal automorphisms of (M, p) of order k.

For a germ of a holomorphic map f at $p \in \mathbb{C}^N$, denote by $j_p^k f$ its k-jet at p. Also denote by $J_p^k(\mathbb{C}^N, \mathbb{C}^N)$ the k-th jet space of holomorphic self maps of $\mathbb{C}^N$ at p. Our goal in this section is to prove the following property that may be of independent interest:

Proposition 3.1. *Let (M, p) be a germ of a real-analytic generic submanifold in $\mathbb{C}^N$ of codimension d. Suppose (M, p) is l-nondegenerate and of type ν. Then for any $k \geq (2d(\nu-1)+2)(2d+3)l$, the image of $\mathsf{aut}^k(M, p)$ under the jet evaluation map*

$$j_p^{(2d+3)l} \colon \mathsf{aut}^k(M, p) \to J_p^{(2d+3)l}(\mathbb{C}^N, \mathbb{C}^N)$$

is a totally real linear subspace of $J_p^{(2d+3)l}(\mathbb{C}^N, \mathbb{C}^N)$.

Proof. Without loss of generality we may assume $p = 0$. Let $L := \sum_j \xi^j \frac{\partial}{\partial z_j} \in \mathsf{aut}^k(M, p)$ be an approximate infinitesimal automorphism and let $\{\theta_t : t \in (-\varepsilon, \varepsilon)\}$ be its local flow defined in a neighborhood of 0 in $\mathbb{C}^N$. Since L is holomorphic, $\theta_t(z)$ is holomorphic in z and real-analytic in t. Consider the power series expansion

$$\theta_t(z) = \sum_{j \geq 0} \Theta_j(z) t^j$$

of $\theta_t(z)$ with respect to t, where $\Theta_j \colon \mathbb{C}^N \to \mathbb{C}^N$ is the germ of a holomorphic map at 0. Then $\Theta_0 = \mathsf{id}$ and $\Theta_1 = (\xi^1, \ldots, \xi^N)$.

Now let $r(z, \overline{z}) = (r^1(z, \overline{z}), \ldots, r^d(z, \overline{z}))$ be a real-analytic defining function of (M, p) and let

$$h(z, \overline{z}, t) := r(\theta_t(z), \overline{\theta_t(z)}).$$

Consider the power series expansion

$$h(z, \overline{z}, t) = \sum_{j \geq 0} h_j(z, \overline{z}) t^j$$

of h with respect to t. Since

$$r(\theta_t(z), \overline{\theta_t(z)}) = r(z, \overline{z}) + 2t\, \mathsf{Re}\, Lr(z, \overline{z}) + o(|t|) \quad \text{as} \quad t \to 0,$$

we have $h_0 \equiv 0$ on M and $h_1(z, \overline{z}) = o(|z|^k)$ as $z \to 0$ in M by the assumptions. Then, by the standard complexification argument, we obtain on the complexification $\mathcal{M} := \{(z, \zeta) \in U \times \overline{U} : r(z, \zeta) = 0\}$ of M, where U is a sufficiently small neighborhood of 0 in $\mathbb{C}^N$,

$$r(\theta_t(z), \overline{\theta}_t(\zeta)) = h(z, \zeta, t) = \sum_{j \geq 1} h_j(z, \zeta) t^j \tag{3.1}$$

such that

$$h_1(z, \zeta) = o(|(z, \zeta)|^k) \quad \text{as} \quad (z, \zeta) \in \mathcal{M} \to 0.$$

Choose a linear basis of real-analytic $(0,1)$ vector fields $L_1, \ldots, L_n$, on M near 0, where $n = N - d$. By a slight abuse of notation we write the same letters for their complexifications on $\mathcal{M}$. For a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, write $|\alpha| := \alpha_1 + \cdots + \alpha_n$ and $L^\alpha := L_1^{\alpha_1} \cdots L_n^{\alpha_n}$. Since M is l-nondegenerate and θ_t is invertible at 0, we have near 0 the following span condition (see [BER99a, Proposition 11.2.4])

$$\mathsf{span}\{L^\alpha r_z^m(\theta_t(z), \overline{\theta_t(z)}) : 1 \leq m \leq d, |\alpha| \leq l\} = \mathbb{C}^N,$$

where $r_z^m := \left(\frac{\partial r^m}{\partial z_1}, \ldots, \frac{\partial r^m}{\partial z_N}\right) \in \mathbb{C}^N$ is the gradient of the mth component of r.

Then by applying the operators L^α for $|\alpha| \leq l$ to (3.1) and using the implicit function theorem, we obtain as in [KZ01, §4.1] the basic reflection identity

$$\theta_t(z) = \Psi(z, \zeta, j_\zeta^l \overline{\theta}_t) + h^0(z, \zeta, t), \quad (z, \zeta) \in \mathcal{M}, \tag{3.2}$$

where Ψ is a holomorphic map defined in a neighborhood of $(0, 0, j_0^l \mathsf{id})$ in $\mathbb{C}^N \times J^l(\mathbb{C}^N, \mathbb{C}^N)$ which is independent of θ_t, and $h^0 = \sum_{j \geq 1} h_j^0(z, \zeta) t^j$ is a holomorphic map (depending on θ_t) defined in a neighborhood of 0 in $\mathbb{C}^N \times \mathbb{C}^N \times (-\varepsilon, \varepsilon)$ such that

$$h_1^0(z, \zeta) = o(|(z, \zeta)|^{k-l}) \quad \text{as} \quad (z, \zeta) \in \mathcal{M} \to 0.$$

Moreover, differentiating (3.2) in z, we obtain for $\tau \leq k - l$,

$$j_z^\tau \theta_t = \Psi^\tau(z, \zeta, j_\zeta^{\tau + l} \overline{\theta}_t) + h^\tau(z, \zeta, t), \quad (z, \zeta) \in \mathcal{M}, \tag{3.3}$$

where $h^\tau(z, \zeta, t) = \sum_{j \geq 1} h_j^\tau(z, \zeta) t^j$ is a holomorphic map satisfying

$$h_1^\tau(z, \zeta) = o(|(z, \zeta)|^{k-(l+\tau)}) \quad \text{as} \quad (z, \zeta) \in \mathcal{M} \to 0.$$

For every positive integer μ, define the iterated complexification $\mathcal{M}^\mu$ of order μ as follows (see [Z97, Z99, KZ01]). Let $\mathcal{M}^\mu$ be the connected component of $\{(\zeta^\mu, \ldots, \zeta^1, \zeta^0) \in \mathbb{C}^{(\mu+1)N} : r_j(\zeta^j, \zeta^{j-1}) = 0, j = 1, \ldots, \mu\}$ containing 0, where

$$r_j(\zeta^j, \zeta^{j-1}) := \begin{cases} r(\zeta^j, \zeta^{j-1}) & \text{if } j \text{ is odd,} \\ \overline{r}(\zeta^j, \zeta^{j-1}) & \text{if } j \text{ is even.} \end{cases}$$

Then by iterating (3.3) $2d + 3$ times and evaluating at $\zeta^0 = 0$, we obtain

$$\theta_t(z) = \hat{\Psi}(z, \mathcal{B}, j_0^{(2d+3)l} \overline{\theta}_t) + \hat{h}(z, \mathcal{B}, t), \quad (z, \mathcal{B}) \in \mathcal{M}^{2d+3} \cap \{\zeta^0 = 0\},$$

where $\hat{\Psi}$ is independent of θ_t, $\mathcal{B} = (\zeta^{2d+2}, \dots, \zeta^0)$ and $\hat{h}(z, \mathcal{B}, t) = \sum_{j\geq 1} \hat{h}_j(z, \mathcal{B})t^j$ is such that

$$\hat{h}_1(z, \mathcal{B}) = o(|(z, \mathcal{B})|^{k-(2d+3)l}), \quad (z, \mathcal{B}) \in \mathcal{M}^{2d+3} \cap \{\zeta^0 = 0\} \to 0.$$

Since M is of type ν, by similar arguments as in [KZ01, §4.2], we obtain a singular jet parametrization

$$\theta_t(z) = \check{\Psi}\Big(\lambda, \frac{z}{\lambda^m}, j_0^{(2d+3)l}\overline{\theta}_t\Big) + \check{h}\Big(\lambda, \frac{z}{\lambda^m}, t\Big)$$

for some integer $m \leq 2d(\nu - 1)$, where $\lambda \in \mathbb{C}$, $\check{\Psi}$ is a holomorphic map in a neighborhood of $(0, 0, j_0^{(2d+3)l}\mathsf{id})$ in $\mathbb{C} \times \mathbb{C}^N \times J_0^{(2d+3)l}(\mathbb{C}^N, \mathbb{C}^N)$ independent of θ_t and $\check{h}(\lambda, \tilde{z}, t) = \sum_{j\geq 1} \check{h}_j(\lambda, \tilde{z})t^j$ is a holomorphic map in a neighborhood of 0 in $\mathbb{C} \times \mathbb{C}^N \times (-\varepsilon, \varepsilon)$ such that

$$\check{h}_1(\lambda, \tilde{z}) = o(|(\lambda, \tilde{z})|^{k-(2d+3)l}) \quad \text{as} \quad (\lambda, \tilde{z}) \to 0.$$

Then by [KZ01, Lemma 4.4], we have

$$\theta_t(z) = \Phi(z, j_0^{(2d+3)l}\overline{\theta}_t) + \tilde{h}(z, t), \tag{3.4}$$

where Φ is a holomorphic map in a neighborhood of $(0, j^{(2d+3)l}\mathsf{id})$ in $\mathbb{C}^N \times J_0^{(2d+3)l}(\mathbb{C}^N, \mathbb{C}^N)$ and $\tilde{h}(z, t) = \sum_{j\geq 1} \tilde{h}_j(z)t^j$ is a holomorphic map in a neighborhood of 0 in $\mathbb{C}^N \times (-\varepsilon, \varepsilon)$ such that

$$\tilde{h}_1(z) = o\big(|z|^{\frac{k-(2d+3)l}{m+1}}\big) \quad \text{as} \quad z \in \mathbb{C}^N \to 0.$$

We differentiate (3.4) in t at $t = 0$ to obtain a jet parametrization

$$\xi(z) = \hat{\Phi}(z, j_0^{(2d+3)l}\overline{\xi}) + o\big(|z|^{\frac{k-(2d+3)l}{m+1}}\big) \quad \text{as} \quad z \in \mathbb{C}^N \to 0, \tag{3.5}$$

where $\xi := (\xi^1, \dots, \xi^N)$ denote the components of the original infinitesimal automorphism L. Since $m \leq 2d(\nu - 1)$ and

$$k \geq (2d(\nu - 1) + 2)(2d + 3)l \geq (m + 2)(2d + 3)l,$$

differentiation of (3.5) $(2d + 3)l$ times in z at $z = 0$ yields

$$j_0^{(2d+3)l}\xi = \check{\Phi}(j_0^{(2d+3)l}\overline{\xi}),$$

where $\check{\Phi}$ is a holomorphic map in a neighborhood of $j_0^{(2d+3)l}\mathsf{id}$ in $J_0^{(2d+3)l}(\mathbb{C}^N, \mathbb{C}^N)$. Hence we have $\zeta = \check{\Phi}(\bar{\zeta})$ for any $\zeta \in j_0^{(2d+3)l}(\mathsf{aut}^k(M, p))$ and therefore $j_0^{(2d+3)l}(\mathsf{aut}^k(M, p)) \subset J_0^{(2d+3)l}(\mathbb{C}^N, \mathbb{C}^N)$ is totally real. □

4. Rigidity properties of CR-families of automorphisms

Our next step in proving Theorems 1.1 and 1.2 consists of establishing rigidity properties for local CR-families of automorphisms given as germs of smooth CR-maps $\varphi\colon (S, a) \times (M, p) \to (M, p)$, where (S, a) and (M, p) are CR-manifolds. By rigidity here we mean the following property:

Proposition 4.1. *Let (S, a) and (M, p) be germs of smooth CR-manifolds that are finitely nondegenerate and of finite type on dense subsets. If $\varphi\colon (S, a) \times (M, p) \to (M, p)$ is a germ of a smooth CR-map such that $\varphi(a, \cdot) = \mathsf{id}$, then $\varphi(s, \cdot) = \mathsf{id}$ for all $s \in S$ sufficiently close to a.*

In fact, it will follow from the proof that the same conclusion holds under the weaker assumption that (S, a) is only minimal (in the sense of TUMANOV [T88]) on a dense subset.

Proof. Let d be the CR-codimension of (M, p), i.e. the codimension of the complex tangent space $T^c_p M$ in $T_p M$. We first assume that M is finitely nondegenerate and of finite type at p (and not only on a dense subset). Choose l such that M is l-nondegenerate at p. It is shown in [KZ01] that, for any invertible jet $(p, \lambda) \in J^{2(d+1)l}(M, M)$, there exists a $J^{2(d+1)l+1}(M, M)$-valued smooth function Φ defined in a neighborhood of (p, λ) such that all germs of smooth CR-diffeomorphisms $f\colon M \to M$ at any $q \in M$ with $(q, j^{2(d+1)l}_q f)$ sufficiently close to (p, λ), satisfy a complete differential system

$$j^{2(d+1)l+1}_x f = \Phi(x, j^{2(d+1)l}_x f)$$

for x sufficiently close to q.

Now choose $X \in \mathsf{aut}(M, p)$ and let $\{\theta_t : t \in (-\varepsilon, \varepsilon)\}$ be its local flow. Then there is a neighborhood U of p in M such that for all $t \in (-\varepsilon, \varepsilon)$, θ_t is well-defined in U and satisfies

$$j^{2(d+1)l+1}_x \theta_t = \Phi(x, j^{2(d+1)l}_x \theta_t) \tag{4.1}$$

for all $x \in U$. By differentiating (4.1) in t at $t = 0$, we obtain a complete differential system

$$j^{2(d+1)l+1}_x X = \Psi(x, j^{2(d+1)l}_x X), \ \ x \in U$$

for X, where Ψ is a $J^{2(d+1)l+1}(M, TM)$-valued smooth function defined in a neighborhood of $(p, j^{2(d+1)l}_p X)$ in the space $J^{2(d+1)l}(M, TM)$ of $2(d+1)l$-jets of vector fields on M. As a consequence, we obtain finite jet determination for infinitesimal CR-automorphisms: if $X, Y \in \mathsf{aut}(M, p)$ and $j^{2(d+1)l}_p X = j^{2(d+1)l}_p Y$, then $X \equiv Y$.

Let $\varphi\colon (S, a) \times (M, p) \to (M, p)$ be a germ of a CR-map satisfying the assumptions of the Proposition 4.1. For any $(1, 0)$ vector field L on S in a neighborhood of a such that $L(a) \neq 0$, define one-parameter families $\{\theta_t : t \in (-\varepsilon, \varepsilon)\}$ and $\{\eta_t : t \in (-\varepsilon, \varepsilon)\}$ of local CR-diffeomorphisms of M by

$$\theta_t(x) := \varphi(\sigma_1(t), x), \quad \eta_t(x) := \varphi(\sigma_2(t), x),$$

where σ_1 and σ_2 are the integral curves of $\mathsf{Re}\, L$ and $\mathsf{Im}\, L$, respectively such that $\sigma_1(0) = \sigma_2(0) = a$. Therefore $\dot\theta_0$ and $\dot\eta_0$ are infinitesimal CR-automorphisms, where the dot denotes the derivative in t. Moreover, by the definition of θ_t and η_t, we have $\dot\theta_0, \dot\eta_0 \in \Gamma(M, T^c M)$. On the other hand,

$$\dot\eta_0 = \varphi_*(\dot\sigma_2(0), x) = \varphi_*(\mathsf{Im}\, L(a), x) = J\varphi_*(\mathsf{Re}\, L(a), x) = J\dot\theta_0,$$

where J is the complex structure on $T^c M$. Hence we obtain an infinitesimal automorphism $X := \dot\theta_0 \in \mathsf{aut}(M, p) \cap \Gamma(M, T^c M)$ such that also $JX \in \mathsf{aut}(M, p)$.

Now set $k := (2d(\nu - 1) + 2)(2d + 3)l$ as in Proposition 3.1, where ν is the type of M at p. By [KZ01, Proposition 3.1], we can choose a neighborhood U of p and a smooth embedding $\psi\colon U \to \mathbb{C}^{n+d}$, $n = \dim_{CR} M$, which is CR of order $k + 1$ at p, i.e. for any $(0, 1)$ vector field L on M defined near p, $L\psi(x) = o(|x - p|^k)$ as $x \to p$, and such that $\psi(p) = 0$ and $\psi(U)$ is a generic real-analytic submanifold of codimension d.

We write $\psi_*(X) = \mathsf{Re}\sum_{j=1}^{n+d} \xi^j \frac{\partial}{\partial z_j}$. Since ψ is CR of order $k+1$ at p, it follows that each ξ^j is CR of order k at 0. Therefore we can choose a holomorphic vector field $\sum_{j=1}^{n+d} \widetilde{\xi}^j \frac{\partial}{\partial z_j} \in \mathsf{aut}^k(\psi(M), 0)$ such that $j_0^k(\xi^1, \dots, \xi^{n+d}) = j_0^k(\widetilde{\xi}^1, \dots, \widetilde{\xi}^{n+d})$. Define $j\colon \mathsf{aut}(M,p) \to J_0^{(2d+3)l}(\mathbb{C}^{n+d}, \mathbb{C}^{n+d})$ by

$$j(X) := j_0^{(2d+3)l}(\widetilde{\xi}^1, \dots, \widetilde{\xi}^{n+d}).$$

Then by the finite jet determination in $\mathsf{aut}(M,p)$ mentioned above and by Proposition 3.1, j is injective and the image $j(\mathsf{aut}(M,p)) \subset J_0^{(2d+3)l}(\mathbb{C}^{n+d}, \mathbb{C}^{n+d})$ is a totally real linear subspace. Moreover, since ψ is CR of order $k+1$ at p, for any $X \in \mathsf{aut}(M,p) \cap \Gamma_p(M, T^cM)$ we have

$$\varphi_*(JX) = J\varphi_*(X) + o(|x-p|^k), \quad x \in M \to p,$$

and therefore $j(JX) = J(j(X))$ since $JX \in \mathsf{aut}(M,p)$. Since $j(\mathsf{aut}(M,p))$ is totally real and j is injective, this implies $X \equiv 0$. In the above notation this means $\dot{\theta}_0(x) \equiv 0$ or equivalently $\varphi_*(\mathsf{Re}\, L(a), x) = 0$. A similar argument applied to $\varphi_b(s,x) := \varphi_b^{-1}(\varphi(s,x))$ for $\varphi_b := \varphi(b, \cdot)$ and $b \in S$ sufficiently close to a shows that $\varphi_*(\mathsf{Re}\, L(b), x) = 0$. Since L is an arbitrary $(1,0)$ vector field on S, it follows that also $\varphi_*(L,x) = \varphi_*(\overline{L}, x) = 0$. Since (S,a) is of finite type, this implies that $\varphi(\cdot, x)$ is constant for every $x \in M$ (sufficiently close to p). (If (S,a) is merely minimal, the same conclusion follows by observing that $\varphi(\cdot, x)$ is constant along CR-curves on S.) Hence we obtain the required conclusion in the case M is finitely nondegenerate and of finite type at p.

To prove the statement in the general case, suppose that the conclusion does not hold for a germ of a smooth CR-map $\varphi\colon (S,a) \times (M,p) \to (M,p)$. Then the partial derivative $\partial_s \varphi(s,x)$ does not vanish at points arbitrary close to (a,p). On the other hand, the above argument implies that the derivative is zero near all minimal points of S that are sufficiently close to a. By the assumption, the minimal points are dense, and hence we reach a contradiction. The proof is complete. □

5. Local splitting of CR-maps; Proof of Theorem 1.1

In this section we prove Theorem 1.1. Let (M,p) be as in Theorem 1.1 and fix a decomposition into irreducible factors

$$(M,p) \;\cong\; (M_1, p_1) \times \cdots \times (M_m, p_m).$$

It is clear that such a decomposition always exists but a priori may not be unique. Let (M', p') be another CR-manifold and

$$(M', p') \;\cong\; (M_1', p_1') \times \cdots \times (M_{m'}', p_{m'}') \tag{5.1}$$

be a corresponding decomposition. Define

$$(\widetilde{M}, \tilde{p}) := (M_1, p_1) \times \cdots \times (M_{m-1}, p_{m-1}).$$

Since M is finitely nondegenerate and of finite type on a dense subset, it follows directly from the definition that the same holds for $\widetilde{M}$ and M_m.

Now let $f = (f^1, \dots, f^{m'})$ be a germ of a smooth CR-diffeomorphism between (M,p) and (M', p'), where f^j is the jth component with respect to the decomposition (5.1). We fix connected representatives for all germs of CR-manifolds and denote them by the same letters. We may then assume that f maps $M = M_1 \times \cdots \times M_m$ diffeomorphically onto an open connected subset $U' \subset M' = M'_1 \times \cdots \times M'_{m'}$. For an open subset $U_m \subset M_m$, define the subsets $U'_j := f^j(\{\tilde{p}\} \times U_m) \subset M'_j$. We also write $f^{-1} = (g^1, \ldots, g^m)\colon U' \to M$.

Lemma 5.1. *If $U_m \subset M_m$ is a sufficiently small neighborhood of p_m, one has $\Pi_{j=1}^{m'} U'_j \subset f(\{\tilde{p}\} \times M_m)$.*

Proof. By the construction,

$$f(\{\tilde{p}\} \times U_m) = \{(f^1(\tilde{p}, v), \cdots, f^{m'}(\tilde{p}, v)) : v \in U_m\} \subset \Pi_{j=1}^{m'} U'_j.$$

Let

$$\pi\colon \underbrace{(M_m, p_m) \times \cdots \times (M_m, p_m)}_{m'} \times (\widetilde{M}, \tilde{p}) \to (\widetilde{M}, \tilde{p})$$

be the germ given by

$$\pi(v^1, \ldots, v^{m'}, z) := \Big(g^1\big(f^1(z, v^1), \ldots, f^{m'}(z, v^{m'})\big), \ldots, g^{m-1}\big(f^1(z, v^1), \ldots, f^{m'}(z, v^{m'})\big)\Big),$$

where $z \in \widetilde{M}$ and $v^j \in M_m$ for $1 \le j \le m'$. Then $\pi(v, \ldots, v, z) \equiv z$ holds for $(z, v) \in M$ near p. By Proposition 4.1, we conclude that $\pi(v^1, \ldots, v^{m'}, z) \equiv z$ for $(v^1, \ldots, v^{m'}) \in M_m \times \cdots \times M_m$ near $(p_m, \ldots, p_m)$ and for $z \in \widetilde{M}$ near $\tilde{p}$. This implies $f^{-1}(f^1(\tilde{p}, v^1), \ldots, f^{m'}(\tilde{p}, v^{m'})) \in \{\tilde{p}\} \times M_m$ for all $v^1, \ldots, v^{m'} \in M_m$ near p_m and the lemma follows. □

We need the following standard lemma proven here for the convenience of the reader.

Lemma 5.2. *Let $S \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ be a smooth submanifold. Denote by $\pi_j\colon S \to \mathbb{R}^{n_j}$, $j = 1, 2$, the canonical projections. Suppose that, for an open subset $U \subset S$, $\pi_1(U) \times \pi_2(U) \subset S$. Then for any $p \in U$ and any sufficiently small neighborhood Ω of p in U, $\pi_j(\Omega)$ is a submanifold of $\mathbb{R}^{n_j}$ for $j = 1, 2$, and Ω is open in $\pi_1(\Omega) \times \pi_2(\Omega)$.*

Proof. The inclusion $\pi_1(U) \times \pi_2(U) \subset S$ implies $d\pi_1(T_pS) \times d\pi_2(T_pS) \subset T_pS$ for every $p \in U$ and therefore $d\pi_1(T_pS) \times d\pi_2(T_pS) = T_pS$. By the semi-continuity of the dimension of each $d\pi_j(T_pS)$, $j = 1, 2$, with respect to p, we conclude that both dimensions are constant and therefore both π_1 and π_2 are of constant rank on U. The required statement follows from the rank theorem. □

In our situation each U'_j is a subset of M'_j for all $j = 1, \ldots, m'$, such that the product $U'_1 \times \cdots \times U'_{m'}$ is contained in a smooth submanifold that is locally CR-equivalent to M_m by Lemma 5.1. Hence it follows from Lemma 5.2 that, if the neighborhood U_m of p_m in M_m is chosen sufficiently small, each U'_j is a smooth submanifold of M'. Moreover, U_m is a CR-submanifold of M in the sense that its tangent subspace intersects the complex tangent space of M along complex subspaces of constant dimension. Then, $U'_1 \times \cdots \times U'_{m'}$ is a CR-submanifold of M' by Lemma 5.1 and hence each $U'_j \subset M'_j$ is a CR-submanifold. We obtain a decomposition of $(M_m, p_m) \cong (U'_1, p'_1) \times \cdots \times (U'_{m'}, p'_{m'})$ into a product of germs of smooth CR-submanifolds. Since, however, (M_m, p_m) was chosen to be irreducible, it is CR-equivalent to (U'_j, p'_j) for some $j \in \{1, \ldots, m'\}$. Without loss of generality, $j = m'$. Then the other factors U'_j are zero-dimensional. We conclude that $f^1(\tilde{p}, \cdot), \ldots, f^{m'-1}(\tilde{p}, \cdot)$

are constant near p_m and that $f^{m'}(\tilde{p},\cdot)\colon (M_m,p_m)\to(M'_{m'},p'_{m'})$ defines a CR-equivalence. By Proposition 4.1, we obtain:

Lemma 5.3. *For $z\in\widetilde{M}$ sufficiently close to $\tilde{p}$, one has $f^{m'}(z,\cdot)\equiv f^{m'}(\tilde{p},\cdot)$.*

Lemma 5.4. *Set $\widetilde{f}:=(f^1,\dots,f^{m'-1})$, $\widetilde{M'}:=\prod_{j=1}^{m'-1}M'_j$ and $\tilde{p}':=(p'_1,\dots,p'_{m'-1})\in\widetilde{M'}$. Then*

(i) *$\widetilde{f}(\cdot,p_m)\colon(\widetilde{M},\tilde{p})\to(\widetilde{M'},\tilde{p}')$ is a CR-diffeomorphism;*

(ii) *for any $z_m\in M_m$ sufficiently close to p_m, one has $\widetilde{f}(\cdot,z_m)\equiv\widetilde{f}(\cdot,p_m)$.*

From Lemmata 5.3 and 5.4 we conclude that f splits into the product $f=\widetilde{h}\times h^m$ with $\widetilde{h}:=\widetilde{f}(\cdot,p_m)\colon(\widetilde{M},\tilde{p})\to(\widetilde{M'},\tilde{p}')$ and $h^m:=f^{m'}(\tilde{p},\cdot)\colon(M_m,p_m)\to(M'_{m'},p'_{m'})$. The proof of Theorem 1.1 is completed by induction on m.

6. Global splitting of CR-maps; Proof of Theorem 1.2

We now turn to the proof of the global decomposition result stated in Theorem 1.2. Let M and $f\colon M\to M'$ be as in Corollary 1.2. Without loss of generality we may assume that M is connected. Fix decompositions

$$M \;\cong\; M_1\times\cdots\times M_m,\quad M' \;\cong\; M'_1\times\cdots\times M'_{m'}$$

into irreducible factors and write $f=(f^1,\dots,f^{m'})$, where f^i is the i-th component of f.

For any point $(p_1,\dots,p_m)\in M$, $p_i\in M_i$, decompose

$$(M_i,p_i)=(M_{i,1},p_{i,1})\times\cdots\times(M_{i,s_i},p_{i,s_i})$$

and

$$(M'_i,p'_i)=(M'_{i,1},p'_{i,1})\times\cdots\times(M'_{i,r_i},p'_{i,r_i})$$

into local irreducible factors, where $p'_i:=f^i(p)$ and let $\pi'_{j,r}$ be the canonical projection of M'_j to $M'_{j,r}$ defined in a small neighborhood of p'_j, where $M'_{j,r}$ is a representative of $(M'_{j,r},p'_{j,r})$.

Now assume that

$$\dim M_m=\max(\dim M_1,\dots,\dim M_m,\dim M'_1,\dots,\dim M_{m'}).$$

Fix $\tilde{p}:=(p_1,\dots,p_{m-1})\in\widetilde{M}:=M_1\times\cdots\times M_{m-1}$. By Theorem 1.1, the germ $f\colon(M,p)\to(M',p')$ can be written as a product of germs of CR-diffeomorphisms $f^{i,s}\colon(M_{i,s},p_{i,s})\to(M'_{j_{i,s},r_{i,s}},p'_{j_{i,s},r_{i,s}})$. Hence there exists arbitrarily small connected open neighborhood Ω of p_m in M_m such that

$$f(\{\tilde{p}\}\times\Omega)=f^1(\{\tilde{p}\}\times\Omega)\times\cdots\times f^{m'}(\{\tilde{p}\}\times\Omega).$$

Moreover we can choose Ω so that for each j, $j=1,\dots,m'$, $h^{j,r}:=\pi'_{j,r}(f^j(\tilde{p},\cdot))$ is well-defined in Ω,

$$f^j(\{\tilde{p}\}\times\Omega)=h^{j,1}(\Omega)\times\cdots\times h^{j,r_j}(\Omega)$$

and there exists a subset $\mathcal{A}_j\subset\{1,\dots,r_j\}$ such that $h^{j,r}$ is constant if $r\in\mathcal{A}_j$ and of maximal rank at every point of Ω if $r\notin\mathcal{A}_j$.

We claim that

$$f(\{\tilde{p}\}\times M_m)=f^1(\{\tilde{p}\}\times M_m)\times\cdots\times f^{m'}(\{\tilde{p}\}\times M_m). \tag{6.1}$$

Indeed, set

$$\mathfrak{G} := \{(x_1, \dots, x_{m'}) \in \underbrace{M_m \times \cdots \times M_m}_{m'} : (f^1(\tilde{p}, x_1), \dots, f^{m'}(\tilde{p}, x_{m'})) \in f(\{\tilde{p}\} \times M_m)\}. \tag{6.2}$$

For any $x \in M_m$, we can choose a neighborhood $\Omega_x \subset M_m$ of x such that

$$f(\{\tilde{p}\} \times \Omega_x) = f^1(\{\tilde{p}\} \times \Omega_x) \times \cdots \times f^{m'}(\{\tilde{p}\} \times \Omega_x).$$

Therefore $\underbrace{(x, \dots, x)}_{m'}$ is an interior point of $\mathfrak{G}$. Let $\mathfrak{G}_x$ be the maximal connected open set consisting of interior points of $\mathfrak{G}$ containing $(x, \dots, x)$ and let $\overline{\mathfrak{G}}_x$ be its closure in $M_m \times \cdots \times M_m$. Then by continuity of f, $\overline{\mathfrak{G}}_x$ is again a subset of $\mathfrak{G}$.

Choose any $(x_1, \dots, x_{m'}) \in \overline{\mathfrak{G}}_x$. Then by (6.2), there is a point $p_m \in M_m$ such that $f(\tilde{p}, p_m) = (f^1(\tilde{p}, x_1), \dots, f^{m'}(\tilde{p}, x_{m'}))$. Let Ω and $\mathcal{A}_1$ be as above and choose a connected neighborhood Ω_1 of x_1 in M_m such that in Ω_1, $h^{1,r}$ is well-defined for all $r = 1, \dots, r_1$,

$$f^1(\{\tilde{p}\} \times \Omega_1) = h^{1,1}(\{\tilde{p}\} \times \Omega_1) \times \cdots \times h^{1,r_1}(\{\tilde{p}\} \times \Omega_1)$$

and there exists a subset $\mathcal{B} \subset \{1, \dots, r_1\}$ such that $h^{1,r}$ is constant if $r \in \mathcal{B}$ and of maximal rank at every point of Ω_1 otherwise.

Since $(x_1, \dots, x_{m'}) \in \overline{\mathfrak{G}}_x$ and $f(\{\tilde{p}\} \times M_m) \subset M'$ is a locally closed submanifold, there exist an open subset $V \subset \Omega_1$ and a point $(y_2, \dots, y_{m'}) \in \underbrace{M_m \times \cdots \times M_m}_{m'-1}$ arbitrarily close to $(x_2, \dots, x_{m'})$ such that

$$f^1(\{\tilde{p}\} \times V) \times f^2(\tilde{p}, y_2) \times \cdots \times f^{m'}(\tilde{p}, y_m) \subset f(\{\tilde{p}\} \times \Omega).$$

Then $h^{1,r}$ is constant in V for $r \in \mathcal{A}_1$. Since V is an open subset of Ω_1 and $h^{1,r}$ is of maximal rank at every point of Ω_1 if $r \notin \mathcal{B}$, this implies $\mathcal{A}_1 \subset \mathcal{B}$. Therefore $h^{1,r}(y) = h^{1,r}(p_m)$ for all $y \in \Omega_1$ if $r \in \mathcal{A}_1$ and hence $f^1(\{\tilde{p}\} \times \Omega_1) \subset f^1(\{\tilde{p}\} \times \Omega)$ if Ω_1 is sufficiently small.

By the same argument as above we can choose neighborhoods $\Omega_j \subset M_m$ of x_j , $j = 2, \dots, m'$, such that $f^j(\{\tilde{p}\} \times \Omega_j) \subset f^j(\{\tilde{p}\} \times \Omega)$ or equivalently

$$f^1(\{\tilde{p}\} \times \Omega_1) \times \cdots \times f^{m'}(\{\tilde{p}\} \times \Omega_{m'}) \subset f(\{\tilde{p}\} \times \Omega).$$

Then $(x_1, \dots, x_{m'})$ is an interior point of $\overline{\mathfrak{G}}_x$. Since $M_m \times \cdots \times M_m$ is a connected set, this implies $\overline{\mathfrak{G}}_x = M_m \times \cdots \times M_m$.

Now we have $f(\{\tilde{p}\} \times M_m) = f^1(\{\tilde{p}\} \times M_m) \times \cdots \times f^{m'}(\{\tilde{p}\} \times M_m)$. Since, by the local splitting property of f proven in Theorem 1.1, each $f^j(\tilde{p}, \cdot)$ is of constant rank, this implies that $f^j(\{\tilde{p}\} \times M_m)$ is a closed CR-submanifold of M'_j. Since M_m is irreducible in the sense of Corollary 1.2, we may assume that $f^j(\{\tilde{p}\} \times M_m)$ is of zero dimension if $j \neq m'$ and M_m is CR-diffeomorphic to $f^{m'}(\{\tilde{p}\} \times M_m)$. Since M_m is of maximal dimension among the irreducible factors of M and M', this implies $f^{m'}(\{\tilde{p}\} \times M_m)$ is an open subset of $M'_{m'}$. Since $f^{m'}(\{\tilde{p}\} \times M_m)$ is also closed in $M'_{m'}$ and $M'_{m'}$ is connected, we have $f^{m'}(\{\tilde{p}\} \times M_m) = M'_{m'}$. Then by local splitting property of f we can show that for any $(q, x) \in M$, $q \in \widetilde{M}$, $x \in M_m$, sufficiently close to $(\tilde{p}, p_m)$, we have $\tilde{f}(q, x) = \tilde{f}(q, p_m)$ and $f^{m'}(q, x) = f^{m'}(\tilde{p}, x)$, where $\tilde{f} := (f^1, \dots, f^{m'-1})$.

Since $(\tilde{p}, p_m)$ is arbitrary and M is connected, f can be written as a product of two CR-diffeomorphisms $\hat{f}\colon \widetilde{M} \to \widetilde{M'}$, and $\hat{g}\colon M_m \to M'_{m'}$, where $\widetilde{M'} := M'_1 \times \cdots \times M'_{m'-1}$. The proof of Theorem 1.2 is completed by induction on m.

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