

Rigidity of CR maps between Shilov boundaries of bounded symmetric domains

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Abstract Our goal is to establish what seems to be the first rigidity result for CR embeddings between Shilov boundaries of bounded symmetric domains of higher rank. The result states that any such CR embedding is the standard linear embedding up to CR automorphisms. Our basic assumption extends precisely the well-known optimal bound for the rank one case. There are no other restrictions on the ranks, in particular, the difficult case when the target rank is larger than the source rank is also allowed.

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1 Introduction

Rigidity phenomena for *holomorphic isometries* into complex space forms go back to Bochner [6] and Calabi [7] and lead to far going and deeper un-

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derstanding of *metric rigidity* between general *bounded symmetric domains* in the work of Mok, Mok-Ng and Ng among others. The reader is referred to the survey by Mok [29] for more details, see also the very recent work by Yuan-Zhang [40]. Other important rigidity phenomena for bounded symmetric domains, such as the strong rigidity of complex structures of their compact quotients have been discovered by Siu [31, 32]. See also further work [8, 23, 34]

On the other hand, the study of *rigidity of holomorphic maps* originated in the work of Poincaré [30] and later Alexander [1] for maps sending one open piece of the sphere into another. It was Webster [38] who first obtained rigidity for holomorphic maps between pieces of spheres of different dimension, proving that any such map between spheres in \mathbb{C}^n and \mathbb{C}^{n+1} is totally geodesic. Further results in this direction are due to Faran [15], Cima-Suffridge [10, 11], Forstneric [16, 17] and Huang [18] who obtained the best known regularity assumption independent of the dimension difference $n' - n$, for CR maps between pieces of spheres in \mathbb{C}^{n+1} and $\mathbb{C}^{n'+1}$ under the assumption $n' < 2n$. See also the recent work of S.S.-T. Yau [39]. Beyond this bound, the rigidity is known to fail as illustrated by the so-called Whitney map (see e.g. Example 1.1 in [13]). (CR maps are closely related to holomorphic ones, see e.g. [3]). We mention the work by Huang-Ji [20], Huang [19] and Huang-Ji-Xu [21, 22] dealing with this more difficult case, where rigidity has to be replaced by the classification of the maps. On the other note, further rigidity phenomena for CR maps between real hypersurfaces and *hyperquadrics* have been discovered by Ebenfelt-Huang and the second author [13, 14], Baouendi-Huang [2], Baouendi-Ebenfelt-Huang [4, 5] and Ebenfelt-Shroff [12].

However, comparing with metric rigidity mentioned above, holomorphic rigidity for maps between *bounded symmetric domains* D and D' of higher rank remains much less understood. If the rank r' of D' does not exceed the rank r of D and both ranks $r, r' \geq 2$, the rigidity of *proper holomorphic maps* $f: D \rightarrow D'$ was conjectured by Mok [27] and proved by Tsai [35], showing that f is necessarily totally geodesic (with respect to the Bergmann metric).

The remaining case $r < r'$ seems to be very hard and only little is known. Tu [36, 37] established holomorphic rigidity respectively in the equidimensional case (when he proves that the map is biholomorphic and hence $r = r'$) and for maps between Cartan type I bounded symmetric domain $D_{p,p-1}$ and $D_{p,p}$ (see below). Finally, Mok [28] proved the nonexistence of proper holomorphic maps between certain pairs of bounded symmetric domains with arbitrary $r' - r$.

The goal of this paper is to take on the rigidity problem for *locally defined CR embeddings* between Shilov boundaries of general Cartan type I bounded symmetric domains $D_{p,q}$ of higher rank. This includes the interesting case

$r < r'$. To the best authors' knowledge all known results on local CR rigidity deal with maps between *real hypersurfaces* and rely heavily on *Tanaka-Chern-Moser* approach [9, 33] and many of them also on *Tanaka-Webster* connection, which is unavailable for Shilov boundaries of higher rank. In this paper we follow a new approach going back to the general Cartan's moving frame method. To compensate for the lack of the power of Tanaka-Chern-Moser normalization, we introduce a sequence of several subsequent adjustments of moving frames reaching further and further normalization conditions. We prove:

Theorem 1.1 *Let f be a smooth CR embedding between open pieces of Shilov boundaries of two bounded symmetric domains $D_{p,q}$, $D_{p',q'}$ of Cartan type I with $q < p$, $q' < p'$. Assume that the rank $q > 1$ and*

$$p' - q' < 2(p - q). \quad (1.1)$$

Then after composing with suitable automorphisms of $D_{p,q}$ and $D_{p',q'}$, f is given by the block matrix

$$z \mapsto \begin{pmatrix} z & 0 \\ 0 & I \\ 0 & 0 \end{pmatrix}.$$

Note that the assumptions $q < p$ and $q' < p'$ exclude precisely the cases of square matrices, where one of the Shilov boundaries is totally real and consequently CR maps are trivial. Furthermore, our basic assumption (1.1) corresponds precisely to the *optimal bound* $n' < 2n$ mentioned before in the rank 1 case ($q = q' = 1$) of maps between spheres, where $n = p - 1$ and $n' = p' - 1$ are the CR dimensions of the spheres.

2 Preliminaries and adapted frames

Throughout this paper we adopt the Einstein summation convention unless mentioned otherwise. However, if two equal indices appear at the same letter, e.g. Φ_a^a , no summation is assumed. We shall also follow the convention that small Greek indices $\alpha, \beta, \gamma, \delta$ run over $\{1, \dots, q\}$, small Latin indices i, j, k, l over $\{1, \dots, n\}$, small Latin indices a, b, c, d over $\{1, \dots, q'\}$ and large Latin indices I, J, K, L over $\{1, \dots, n'\}$.

Recall that $D_{p,q}$ has the standard realization in the space $\mathbb{C}^{p \times q}$ of $p \times q$ matrices, given by

$$D_{p,q} := \{z \in \mathbb{C}^{p \times q} : I_q - z^* z \text{ is positive definite}\},$$

where I_q is the identity $q \times q$ matrix and $z^* = \bar{z}^t$. The Shilov boundary of $D_{p,q}$ is given by

$$S_{p,q} = \{z \in \mathbb{C}^{p \times q} : I_q - z^* z = 0\}.$$

In particular, $S_{p,q}$ is a symmetric CR manifold of CR dimension $(p - q) \times q$ in the terminology of [24]. For $q = 1$, $S_{p,1}$ is the unit sphere in \mathbb{C}^p . We shall always assume $p > q$ so that $S_{p,q}$ has positive CR dimension, i.e. not totally real.

Example 2.1 The following generalization of the well-known Whitney map

$$\begin{pmatrix} z_{11} & \cdots & z_{1q} \\ \vdots & \ddots & \vdots \\ z_{p1} & \cdots & z_{pq} \end{pmatrix} \mapsto \begin{pmatrix} z_{11} & \cdots & z_{1q'} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{p-1,1} & \cdots & z_{p-1,q'} & 0 & \cdots & 0 \\ z_{11}z_{p1} & \cdots & z_{11}z_{pq'} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{p1}z_{p1} & \cdots & z_{p1}z_{pq'} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix} \in \mathbb{C}^{(p+m) \times (q'+m)} \quad (2.1)$$

restricts to a CR map between the Shilov boundaries, where $1 \leq q' \leq q$ and m is arbitrary. This map is not injective in $D_{p,q}$ and hence is not linear after composing with any automorphisms of $D_{p,q}$ and $D'_{p',q'}$. For $q = q' = 1$, $m = 0$, this is the classical Whitney proper map between unit balls in \mathbb{C}^p and \mathbb{C}^{2p-1} respectively, which corresponds to the equality in (1.1) showing that the latter is an optimal bound.

Example 2.2 The following examples show that there are lots of CR maps between Shilov boundaries for any choices of ranks q and q' . Fix a collection of proper maps $\varphi_1, \dots, \varphi_{q'}$ from the unit ball in \mathbb{C}^p into unit balls in $\mathbb{C}^{m_1}, \dots, \mathbb{C}^{m_{q'}}$ respectively for any choice of integers $m_1, \dots, m_{q'}$. For any q , and any choice of integers $j_1, \dots, j_{q'} \in \{1, \dots, q\}$, define

$$\Phi : \mathbb{C}^{p \times q} \rightarrow \mathbb{C}^{(m_1 + \cdots + m_{q'}) \times q'},$$

such that $\Phi(Z)$ is the block-diagonal matrix with entries $\varphi_1(z_{j_1}), \dots, \varphi_{q'}(z_{j_{q'}})$ on the diagonal. Then Φ restricts to a CR map between Shilov boundaries of the corresponding bounded symmetric domains.

Let $\text{Aut}(S_{p,q})$ be the Lie group of all CR automorphisms of $S_{p,q}$. By [24, Theorem 8.5], every $\varphi \in \text{Aut}(S_{p,q})$ extends to a biholomorphic automorphism of the bounded symmetric domain $D_{p,q}$. Consider the standard linear inclusion

$$z \mapsto \begin{pmatrix} I_q \\ z \end{pmatrix}, \quad z \in S_{p,q}.$$

Then we may regard $S_{p,q}$ as a real submanifold in the Grassmannian $Gr(q, p+q)$ of all q -planes in \mathbb{C}^{p+q} and $\text{Aut}(S_{p,q}) = \text{Aut}(D_{p,q})$ becomes a subgroup of the automorphism group of $Gr(q, p+q)$. In this section we will construct a frame bundle over $S_{p,q}$ associated with the CR structure of $S_{p,q}$ using Grassmannian frames of $Gr(q, p+q)$.

As before, consider the *partial CR dimension* $n = p - q$. The actual CR dimension of $S_{p,q}$ is $(p - q)q = nq$ and $q = r$ is the rank of the bounded symmetric domain $D_{p,q}$.

For column vectors $u = (u_1, \dots, u_{p+q})^t$ and $v = (v_1, \dots, v_{p+q})^t$ in \mathbb{C}^{p+q} , define the Hermitian inner product by

$$\langle u, v \rangle := -(u_1 \bar{v}_1 + \dots + u_q \bar{v}_q) + (u_{q+1} \bar{v}_{q+1} + \dots + u_{p+q} \bar{v}_{p+q}).$$

A *Grassmannian frame adapted to $S_{p,q}$* , or simply *$S_{p,q}$ -frame* is a frame $\{Z_1, \dots, Z_{p+q}\}$ of \mathbb{C}^{p+q} with $\det(Z_1, \dots, Z_{p+q}) = 1$ such that

$$\langle Z_\alpha, Z_{q+n+\beta} \rangle = \langle Z_{q+n+\beta}, Z_\alpha \rangle = \delta_{\alpha\beta}, \quad \langle Z_{q+j}, Z_{q+k} \rangle = \delta_{jk} \quad (2.2)$$

and

$$\langle Z_\Lambda, Z_\Gamma \rangle = 0 \text{ otherwise,} \quad (2.3)$$

where the capital Greek indices Λ, Γ, Ω etc. run from 1 to $p+q$. We also use the notation

$$\begin{aligned} Z &:= (Z_1, \dots, Z_q), & X &:= (X_1, \dots, X_n) := (Z_{q+1}, \dots, Z_{q+n}), \\ Y &:= (Y_1, \dots, Y_q) := (Z_{q+n+1}, \dots, Z_{p+q}), \end{aligned}$$

so that (2.2) can be rewritten as

$$\langle Z_\alpha, Y_\beta \rangle = \langle Y_\beta, Z_\alpha \rangle = \delta_{\alpha\beta}, \quad \langle X_j, X_k \rangle = \delta_{jk}, \quad (2.4)$$

i.e. the scalar product $\langle \cdot, \cdot \rangle$ in basis (Z_α, X_j, Y_β) is given by the matrix

$$\begin{pmatrix} 0 & 0 & I_q \\ 0 & I_n & 0 \\ I_q & 0 & 0 \end{pmatrix}.$$

Let $\mathcal{B}_{p,q}$ be the set of all $S_{p,q}$ -frames. Then $\mathcal{B}_{p,q}$ can be identified with $SU(p, q)$ by the left action. The Maurer-Cartan form $\pi = (\pi_\Lambda^\Gamma)$ on $\mathcal{B}_{p,q}$ is given by the equation

$$dZ_\Lambda = \pi_\Lambda^\Gamma Z_\Gamma, \quad (2.5)$$

where π satisfies the trace-free condition

$$\sum_\Lambda \pi_\Lambda^\Lambda = 0$$

and the structure equation

$$d\pi_\Lambda^\Gamma = \pi_\Lambda^\Omega \wedge \pi_\Omega^\Gamma. \quad (2.6)$$

More explicitly, using the block matrix representation with respect to the basis (Z, X, Y) , we can write

$$\pi = \begin{pmatrix} \pi_\alpha^\beta & \pi_\alpha^{q+j} & \pi_\alpha^{q+n+\beta} \\ \pi_{q+k}^\beta & \pi_{q+k}^{q+j} & \pi_{q+k}^{q+n+\beta} \\ \pi_{q+n+\alpha}^\beta & \pi_{q+n+\alpha}^{q+j} & \pi_{q+n+\alpha}^{q+n+\beta} \end{pmatrix} =: \begin{pmatrix} \psi_\alpha^\beta & \theta_\alpha^j & \varphi_\alpha^\beta \\ \sigma_k^\beta & \omega_k^j & \theta_k^\beta \\ \xi_\alpha^\beta & \sigma_\alpha^j & \widehat{\psi}_\alpha^\beta \end{pmatrix}, \quad (2.7)$$

which satisfies the symmetry relations

$$\begin{pmatrix} \psi_\alpha^\beta & \theta_\alpha^j & \varphi_\alpha^\beta \\ \sigma_k^\beta & \omega_k^j & \theta_k^\beta \\ \xi_\alpha^\beta & \sigma_\alpha^j & \widehat{\psi}_\alpha^\beta \end{pmatrix} = - \begin{pmatrix} \widehat{\psi}_{\bar{\beta}}^{\bar{\alpha}} & \theta_{\bar{j}}^{\bar{\alpha}} & \varphi_{\bar{\beta}}^{\bar{\alpha}} \\ \sigma_{\bar{\beta}}^{\bar{k}} & \omega_{\bar{j}}^{\bar{k}} & \theta_{\bar{\beta}}^{\bar{k}} \\ \xi_{\bar{\beta}}^{\bar{\alpha}} & \sigma_{\bar{j}}^{\bar{\alpha}} & \psi_{\bar{\beta}}^{\bar{\alpha}} \end{pmatrix} \quad (2.8)$$

that follow directly by differentiating (2.2).

The structure equations (2.6) can be rewritten as

$$d\varphi_\alpha^\beta = \psi_\alpha^\gamma \wedge \varphi_\gamma^\beta + \theta_\alpha^l \wedge \theta_l^\beta + \varphi_\alpha^\gamma \wedge \widehat{\psi}_\gamma^\beta \quad (2.9)$$

$$d\theta_\alpha^j = \psi_\alpha^\gamma \wedge \theta_\gamma^j + \theta_\alpha^l \wedge \omega_l^j + \varphi_\alpha^\gamma \wedge \sigma_\gamma^j \quad (2.10)$$

$$d\psi_\alpha^\beta = \psi_\alpha^\gamma \wedge \psi_\gamma^\beta + \theta_\alpha^l \wedge \sigma_l^\beta + \varphi_\alpha^\gamma \wedge \xi_\gamma^\beta \quad (2.11)$$

$$d\omega_k^j = \sigma_k^\gamma \wedge \theta_\gamma^j + \omega_k^l \wedge \omega_l^j + \theta_k^\gamma \wedge \sigma_\gamma^j \quad (2.12)$$

$$d\sigma_k^\beta = \sigma_k^\gamma \wedge \psi_\gamma^\beta + \omega_k^l \wedge \sigma_l^\beta + \theta_k^\gamma \wedge \xi_\gamma^\beta \quad (2.13)$$

$$d\xi_\alpha^\beta = \xi_\alpha^\gamma \wedge \psi_\gamma^\beta + \sigma_\alpha^l \wedge \sigma_l^\beta + \widehat{\psi}_\alpha^\gamma \wedge \xi_\gamma^\beta, \quad (2.14)$$

in particular,

$$d\varphi_\alpha^\beta = \theta_\alpha^j \wedge \theta_j^\beta \mod \varphi,$$

where φ is the span of φ_α^β for all α, β .

By abuse of notation, we also denote by Z the q -dimensional subspace of \mathbb{C}^{p+q} spanned by Z_1, \dots, Z_q . Hence Z represents a point in $S_{p,q}$ and vice versa, any point in $S_{p,q}$ is represented by Z corresponding to an adapted frame (Z, X, Y) . Then $\mathcal{B}_{p,q}$ can be regarded as a bundle over $S_{p,q}$ via the projection map $(Z, X, Y) \rightarrow Z$. By another abuse of notation, we shall also use the same letters for the components of π and their pullbacks to $S_{p,q}$ via a fixed section. Note that fixing a section means precisely choosing an adapted frame (Z, X, Y) at every point x of (an open subset of) $S_{p,q}$ such that Z represents x as a point in the Grassmannian.

The defining equations of $S_{p,q}$ can be written as

$$S_{p,q} = \{[V] \in Gr(q, p+q) : \langle \cdot, \cdot \rangle|_V = 0\}$$

and hence their differentiation yields

$$\langle dZ_\Lambda, Z_\Gamma \rangle + \langle Z_\Lambda, dZ_\Gamma \rangle = 0. \quad (2.15)$$

By substituting $dZ_\Lambda = \pi_\Lambda^\Gamma Z_\Gamma$ into $(1, 0)$ component of (2.15) we obtain, in particular,

$$\varphi_\alpha^\gamma \langle Y_\gamma, Z_\beta \rangle = \varphi_\alpha^\beta = 0,$$

when restricted to the $(1, 0)$ tangent space. Comparing the dimensions, we conclude that the kernel of $\{\varphi_\alpha^\beta, \alpha, \beta = 1, \dots, q\}$ forms the CR bundle of $S_{p,q}$, i.e.,

$$\ker(\varphi|_Z) = T_Z^{1,0} S_{p,q} \oplus T_Z^{0,1} S_{p,q}.$$

In other words, $\varphi = (\varphi_\alpha^\beta)$ span the space of contact forms on $S_{p,q}$. Since

$$dZ_\alpha = \psi_\alpha^\beta Z_\beta + \varphi_\alpha^\beta Y_\beta + \theta_\alpha^j X_j$$

and $\varphi = (\varphi_\alpha^\beta)$ is a contact form at $Z = (Z_1, \dots, Z_q)$, we conclude that φ_α^β and θ_α^j form together a basis in the space of all $(1, 0)$ forms.

For a change of frame given by

$$\begin{pmatrix} \tilde{Z} \\ \tilde{X} \\ \tilde{Y} \end{pmatrix} := U \begin{pmatrix} Z \\ X \\ Y \end{pmatrix},$$

π changes via

$$\tilde{\pi} = dU \cdot U^{-1} + U \cdot \pi \cdot U^{-1}.$$

There are several types of frame changes.

Definition 2.3 We call a change of frame

(i) change of position if

$$\tilde{Z}_\alpha = W_\alpha^\beta Z_\beta, \quad \tilde{Y}_\alpha = V_\alpha^\beta Y_\beta, \quad \tilde{X}_j = X_j,$$

where $W = (W_\alpha^\beta)$ and $V = (V_\alpha^\beta)$ are $q \times q$ matrices satisfying $V^*W = I_q$;

(ii) change of real vectors if

$$\tilde{Z}_\alpha = Z_\alpha, \quad \tilde{X}_j = X_j, \quad \tilde{Y}_\alpha = Y_\alpha + H_\alpha^\beta Z_\beta,$$

or

$$\begin{pmatrix} \tilde{Z}_\alpha \\ \tilde{X}_j \\ \tilde{Y}_\alpha \end{pmatrix} = \begin{pmatrix} I_q & 0 & 0 \\ 0 & I_n & 0 \\ H_\alpha^\beta & 0 & I_q \end{pmatrix} \begin{pmatrix} Z_\beta \\ X_k \\ Y_\beta \end{pmatrix}, \quad (2.16)$$

where $H = (H_\alpha^\beta)$ is a hermitian matrix;

(iii) dilation if

$$\tilde{Z}_\alpha = \lambda_\alpha^{-1} Z_\alpha, \quad \tilde{Y}_\alpha = \lambda_\alpha Y_\alpha, \quad \tilde{X}_j = X_j,$$

where $\lambda_\alpha > 0$;

(iv) rotation if

$$\tilde{Z}_\alpha = Z_\alpha, \quad \tilde{Y}_\alpha = Y_\alpha, \quad \tilde{X}_j = U_j^k X_k,$$

where (U_j^k) is a unitary matrix.

Consider a change of position as in Definition 2.3. Then φ and θ change to

$$\tilde{\varphi}_\alpha^\beta = W_\alpha^\gamma \varphi_\gamma^\delta W_\delta^{*\beta}, \quad W_\delta^{*\beta} = \overline{W_\beta^\delta}, \quad \tilde{\theta}_\alpha^j = W_\alpha^\beta \theta_\beta^j.$$

We shall also make use of the change of frame given by

$$\tilde{Z}_\alpha = Z_\alpha, \quad \tilde{X}_j = X_j + C_j^\beta Z_\beta, \quad \tilde{Y}_\alpha = Y_\alpha + A_\alpha^\beta Z_\beta + B_\alpha^j X_j,$$

or

$$\begin{pmatrix} \tilde{Z}_\alpha \\ \tilde{X}_j \\ \tilde{Y}_\alpha \end{pmatrix} = \begin{pmatrix} I_q & 0 & 0 \\ C_j^\beta & I_n & 0 \\ A_\alpha^\beta & B_\alpha^j & I_q \end{pmatrix} \begin{pmatrix} Z_\beta \\ X_k \\ Y_\beta \end{pmatrix}, \quad (2.17)$$

such that

$$C_j^\alpha + B_j^\alpha = 0$$

and

$$(A_\alpha^\beta + \overline{A_\beta^\alpha}) + B_\alpha^j B_j^\beta = 0,$$

where

$$B_j^\alpha := \overline{B_\alpha^j}.$$

Then the new frame $(\tilde{Z}, \tilde{Y}, \tilde{X})$ is an $S_{p,q}$ -frame. In fact,

$$\begin{aligned} 0 &= \langle \tilde{Y}_\alpha, \tilde{Y}_\beta \rangle = \langle Y_\alpha + A_\alpha^\delta Z_\delta + B_\alpha^j X_j, Y_\beta + A_\beta^\gamma Z_\gamma + B_\beta^k X_k \rangle \\ &= A_\alpha^\beta \langle Z_\beta, Y_\beta \rangle + \overline{A_\beta^\alpha} \langle Y_\alpha, Z_\alpha \rangle + \sum_j B_\alpha^j \overline{B_\beta^j} \langle X_j, X_j \rangle \\ &= (A_\alpha^\beta + \overline{A_\beta^\alpha}) + \sum_j B_\alpha^j \overline{B_\beta^j}, \end{aligned} \quad (2.18)$$

and

$$\begin{aligned} 0 &= \langle \tilde{X}_j, \tilde{Y}_\alpha \rangle = \langle X_j + C_j^\beta Z_\beta, Y_\alpha + A_\alpha^\delta Z_\delta + B_\alpha^k X_k \rangle \\ &= C_j^\alpha \langle Z_\alpha, Y_\alpha \rangle + \overline{B_\alpha^j} \langle X_j, X_j \rangle \\ &= C_j^\alpha + \overline{B_\alpha^j}, \end{aligned} \quad (2.19)$$

whereas the other scalar products are obviously zero. Furthermore, we claim that the related 1-forms $\tilde{\varphi}_\alpha^\beta$ remain the same, while $\tilde{\theta}_\alpha^j$ change to

$$\tilde{\theta}_\alpha^j = \theta_\alpha^j - \varphi_\alpha^\beta B_\beta^j.$$

Indeed, differentiation yields

$$\begin{aligned} d\tilde{Z}_\alpha &= \tilde{\psi}_\alpha^\beta \tilde{Z}_\beta + \tilde{\theta}_\alpha^j \tilde{X}_j + \tilde{\varphi}_\alpha^\beta \tilde{Y}_\beta \\ &= \tilde{\psi}_\alpha^\beta Z_\beta + \tilde{\theta}_\alpha^j (X_j + C_j^\beta Z_\beta) + \tilde{\varphi}_\alpha^\beta (Y_\beta + A_\beta^\gamma Z_\gamma + B_\beta^j X_j) \\ &= dZ_\alpha = \psi_\alpha^\beta Z_\beta + \theta_\alpha^j X_j + \varphi_\alpha^\beta Y_\beta \end{aligned}$$

and the claim follows from identifying the coefficients.

3 Cartan's Lemma

We shall routinely use the Cartan's Lemma for complex-valued forms:

Lemma 3.1 (Cartan's Lemma) *Let $\theta_1, \dots, \theta_r$ be complex-linearly independent complex-valued 1-forms on a real manifold M and $\varphi_1, \dots, \varphi_r$ be further*

complex-valued 1-forms on M satisfying

$$\theta_1 \wedge \varphi_1 + \cdots + \theta_r \wedge \varphi_r = 0. \quad (3.1)$$

Then

$$\varphi_j = 0 \pmod{\{\theta_1, \dots, \theta_r\}}$$

for each $j = 1, \dots, r$.

Proof Complete $\theta_1, \dots, \theta_r$ to a basis $\theta_1, \dots, \theta_s$ in the space of all complex-valued 1-forms on M . Then we can write

$$\varphi_j = c_j^k \theta_k$$

for suitable coefficients c_j^k . Then substituting into (3.1), using the fact that the set of $\theta_i \wedge \theta_j$ with $i < j$ is a basis in the space of all 2-forms, and identifying coefficients of $\theta_j \wedge \theta_k$ for $j \leq r, k > r$, we conclude

$$c_j^k = 0, \quad j \leq r < k,$$

and the claim follows. \square

4 Determination of Φ_a^b and Θ_a^J modulo φ using the Levi form identities

Let $p > q$, $p' > q'$ be positive integers and let f be a local CR embedding from $S_{p,q}$ into $S_{p',q'}$. Denote by $\mathcal{B}_{p,q}$ and $\mathcal{B}_{p',q'}$ the Grassmannian frame bundles adapted to $S_{p,q}$ and $S_{p',q'}$ respectively. We set $n := p - q$, $n' := p' - q'$ and follow the index convention at the beginning of Sect. 2.

We shall consider the connection forms φ_α^β , θ_α^j , ψ_α^β , ω_j^k , σ_j^β , ξ_α^β on $\mathcal{B}_{p,q}$ pulled back to $S_{p,q}$ and denote by capital letters Φ_a^b , Θ_a^J , Ψ_a^b , Ω_J^K , Σ_K^b , Ξ_a^b their corresponding counterparts on $\mathcal{B}_{p',q'}$ pulled back to $S_{p',q'}$. Furthermore, we shall adopt the convention that any form is assumed to be zero whenever its indices are out of the range where the form is defined, e.g. $\theta_a^J = 0$ if either $a > q$ or $J > n$, or $\varphi_a^b = 0$ if either $a > q$ or $b > q$.

Since $\varphi = (\varphi_\alpha^\beta)$ and $\Phi = (\Phi_a^b)$ are contact forms on $S_{p,q}$ and $S_{p',q'}$, respectively, the pull back of Φ via f is a linear combination of $\varphi = (\varphi_\alpha^\beta)$.

We shall abuse the notation by writing Σ instead of $f^*\Sigma$ for any form Σ on $S_{p',q'}$. Thus all our forms will be understood on $S_{p,q}$ and any form on $S_{p',q'}$ will be assumed pulled back to $S_{p,q}$ via the given CR map f without explicit mentioning.

In this section our analysis will be based on using the structure equation for φ modulo the ideal generated by the contact forms φ_α^β , i.e. on the equations

$$d\varphi_a^b = \theta_a^j \wedge \theta_j^b \mod \varphi, \quad d\Phi_a^b = \Theta_\alpha^J \wedge \Theta_J^b \mod \varphi. \quad (4.1)$$

By writing identities modulo φ we shall always mean that the difference between the left- and right-hand sides is contained in the ideal generated by the components φ_α^β in the exterior algebra. In the second identity we have also used the fact mentioned above that (the pullback of) any Φ_a^b is a linear combination of φ_α^β . Note that due to our convention, both sides of the first equation are zero if either $a > q$ or $b > q$ and for the same reason the summation is only performed over $j \in \{1, \dots, n\}$.

4.1 Determination of Φ_1^1

Consider the diagonal terms Φ_a^a , $a = 1, \dots, q'$. Suppose that (the pullbacks of) Φ_a^a vanish identically for all a . Then (4.1) yields

$$0 = d\Phi_a^a = - \sum_J \Theta_a^J \wedge \overline{\Theta_a^J} \mod \varphi.$$

Since each Θ_a^J is a $(1, 0)$ form and each wedge product is non-negative on (T, \bar{T}) where T is any $(1, 0)$ vector, it follows that

$$\Theta_a^J = 0 \mod \varphi,$$

which contradicts the assumption that f is an embedding.

Hence there exists at least one diagonal term of Φ whose pullback does not vanish identically. Choose such a diagonal term of Φ , say Φ_1^1 . Then on an open set, $\Phi_1^1 \neq 0$. Since the pullback of Φ_1^1 to $S_{p,q}$ is a contact form, we can write

$$\Phi_1^1 = c_\alpha^\beta \varphi_\beta^\alpha$$

for some smooth functions c_α^β . Since (φ_α^β) and (Φ_a^b) are antihermitian, the matrix (c_α^β) is hermitian. Then there exists a change of frame on $S_{p,q}$ (change of position in Definition 2.3) given by

$$\tilde{Z}_\alpha = U_\alpha^\beta Z_\beta, \quad \tilde{Y}_\alpha = U_\alpha^\beta Y_\beta, \quad \tilde{X}_j = X_j,$$

for some unitary matrix U such that c_α^β is diagonalized and hence the new contact forms φ_α^β , $\alpha, \beta = 1, \dots, q$, satisfy

$$\Phi_1^1 = \sum_{\alpha=1}^r c_\alpha \varphi_\alpha^\alpha, \quad 1 \leq r \leq q,$$

where c_α , $\alpha = 1, \dots, r$, are nonzero real valued smooth functions. Then (4.1) yields

$$\sum_J \Theta_1^J \wedge \overline{\Theta_1^J} = \sum_{\alpha, j} c_\alpha \theta_\alpha^j \wedge \overline{\theta_\alpha^j} \mod \varphi, \quad (4.2)$$

which implies $c_\alpha > 0$ in view of the non-negativity mentioned above since the forms $\theta_\alpha^j \wedge \overline{\theta_\alpha^j}$ are linearly independent. Hence after dilation of Φ_1^1 , we may assume that

$$c_1 = 1.$$

Lemma 4.1 *Assuming $n' < 2n$, we have $r = 1$ and*

$$\Phi_1^1 = \varphi_1^1, \quad (4.3)$$

$$\Theta_1^J = \theta_1^J \mod \varphi. \quad (4.4)$$

Proof Let

$$\Theta_1^J = h_j^{J, \alpha} \theta_\alpha^j \mod \varphi. \quad (4.5)$$

Then (4.2) implies

$$\sum_J h_j^{J, \alpha} \overline{h_k^{J, \beta}} = c_\alpha \delta_{\alpha\beta} \cdot \delta_{jk},$$

where $c_\alpha := 0$ for $\alpha > r$. Thus the vectors $h_j^\alpha := (h_j^{1, \alpha}, \dots, h_j^{n', \alpha})$ are pairwise orthogonal and have length c_α independent of j . Therefore after a suitable rotation (see Definition 2.3)

$$\tilde{\Theta}_a^J = \Theta_a^K U_K^J,$$

where (U_K^J) is unitary, we may assume that h_j^1 , whose length is $c_1 = 1$, are precisely the first n standard unit vectors in $\mathbb{C}^{n'}$, i.e.

$$h_j^{J, 1} = \delta_{Jj}. \quad (4.6)$$

Then for every fixed $\alpha \neq 1$, we have n orthogonal vectors h_j^α in the span of the last $n' - n$ standard unit vectors. Since $n' - n < n$ by our assumption, the latter is only possible when $h_j^\alpha = 0$ for all $\alpha \neq 1$. Thus we obtain

$$h_j^{J, \alpha} = \delta_{Jj} \delta_{1\alpha}. \quad (4.7)$$

Then (4.5) implies (4.4) and hence (4.2) implies $r = 1$ and therefore (4.1) implies (4.3). \square

4.2 Determination of Φ_2^2 and Φ_2^1

Consider the ideal θ_α generated by θ_α^j for $1 \leq j \leq n$. Let

$$\Phi_a^1 = \lambda_a \varphi_1^1 \mod \{\varphi_\alpha^\beta, \alpha \geq 2 \text{ or } \beta \geq 2\}, \quad a \geq 2, \quad (4.8)$$

for some smooth functions λ_a , $a = 2, \dots, q'$. Then (4.1) together with Lemma 4.1 imply

$$\Theta_a^j \wedge \theta_j^1 = \lambda_a \theta_1^j \wedge \theta_j^1 \mod \{\theta_\alpha, \overline{\theta_\alpha}, \alpha \geq 2\}, \quad \varphi, \quad a \geq 2. \quad (4.9)$$

Then there exists a change of position that leaves Θ_1^J invariant and replaces Θ_a^J with $\Theta_a^J - \lambda_a \Theta_1^J$, $a \geq 2$, (see the discussion after Definition 2.3). The same change of position leaves Φ_1^1 invariant and transforms Φ_a^1 into $\Phi_a^1 - \lambda_a \Phi_1^1$ for $a \geq 2$. After performing such change of position, (4.8) becomes

$$\Phi_a^1 = 0 \mod \{\varphi_\alpha^\beta : \alpha \geq 2 \text{ or } \beta \geq 2\}, \quad a \geq 2,$$

and (4.9) becomes

$$\Theta_a^j \wedge \theta_j^1 = 0 \mod \{\theta_\alpha, \overline{\theta_\alpha} : \alpha \geq 2\}, \quad \varphi, \quad a \geq 2. \quad (4.10)$$

Since Θ_a^j are $(1, 0)$ but θ_j^1 are $(0, 1)$ and linearly independent, it follows that

$$\Theta_a^j = 0 \mod \{\theta_\alpha : \alpha \geq 2\}, \quad \varphi, \quad a \geq 2. \quad (4.11)$$

Next for each $a \geq 2$, let

$$\Phi_a^a = \lambda_{a,\beta} \varphi_1^\beta \mod \{\varphi_\alpha^\gamma : \alpha \geq 2\} \quad (4.12)$$

for some functions $\lambda_{a,\beta}$. Suppose there exists a and β such that $\lambda_{a,\beta} \neq 0$. We may assume $a = 2$. Using the identity

$$d\Phi_2^2 = \Theta_2^J \wedge \Theta_J^2 \mod \Phi$$

together with (4.11) we obtain

$$\sum_{J=n+1}^{n'} \Theta_2^J \wedge \Theta_J^2 = \lambda_{2,\beta} \theta_1^j \wedge \theta_j^\beta \mod \{\theta_\alpha : \alpha \geq 2\}, \quad \varphi, \quad (4.13)$$

where $\lambda_{2,\beta} \neq 0$. On the left-hand side we have a linear combination of $n' - n$ $(1, 0)$ forms, whereas on the right-hand side we have a linear combination of at least n linear independent $(1, 0)$ forms with nonzero coefficients. Since $n' -$

$n < n$, this is impossible. Hence we have $\lambda_{a,\beta} = 0$ for all a, β and therefore (4.12) implies

$$\Phi_a^a = 0 \mod \{\varphi_\alpha^\beta : \alpha \geq 2\}, \quad a \geq 2.$$

Since Φ_a^b and φ_α^β are antihermitian, we also have

$$\Phi_a^a = 0 \mod \{\varphi_\alpha^\beta : \beta \geq 2\}, \quad a \geq 2,$$

and hence

$$\Phi_a^a = 0 \mod \{\varphi_\alpha^\beta : \alpha, \beta \geq 2\}, \quad a \geq 2. \quad (4.14)$$

Now (4.1) implies

$$\sum_{J=n+1}^{n'} \Theta_a^J \wedge \overline{\Theta_a^J} = 0 \mod \{\theta_\alpha : \alpha \geq 2\}, \quad \varphi, \quad a \geq 2, \quad (4.15)$$

which implies

$$\Theta_a^J = 0 \mod \{\theta_\alpha : \alpha \geq 2\}, \quad \varphi, \quad a \geq 2, \quad J > n. \quad (4.16)$$

Together with (4.11) this yields

$$\Theta_a^J = 0 \mod \{\theta_\alpha : \alpha \geq 2\}, \quad \varphi, \quad a \geq 2. \quad (4.17)$$

Now we redo our procedure for Φ_a^b . We can write

$$\Phi_a^b = \lambda_a^{b\alpha} \varphi_\alpha^\beta \quad (4.18)$$

for which (4.1) yields

$$\Theta_a^J \wedge \Theta_J^b = \lambda_a^{b\alpha} \theta_\alpha^j \wedge \theta_j^\beta \mod \varphi.$$

Then substituting (4.17) we obtain

$$\lambda_a^{b\alpha} \theta_\alpha^j \wedge \theta_j^\beta = 0 \mod \{\theta_\gamma^k \wedge \theta_l^\delta : \gamma, \delta \geq 2\}, \quad \varphi, \quad a, b \geq 2, \quad (4.19)$$

which implies

$$\lambda_a^{b\alpha} = \lambda_a^{b1} = 0, \quad a, b \geq 2. \quad (4.20)$$

Hence (4.18) yields

$$\Phi_a^b = 0 \mod \{\varphi_\alpha^\beta : \alpha, \beta \geq 2\}, \quad a, b \geq 2. \quad (4.21)$$

Summarizing we obtain the following:

$$\Phi_a^1 = 0 \pmod{\{\varphi_\alpha^\beta : \alpha \geq 2 \text{ or } \beta \geq 2\}}, \quad a \geq 2, \quad (4.22)$$

$$\Phi_a^b = 0 \pmod{\{\varphi_\alpha^\beta : \alpha, \beta \geq 2\}}, \quad a, b \geq 2, \quad (4.23)$$

$$\Theta_a^J = 0 \pmod{\{\theta_\alpha : \alpha \geq 2\}, \varphi}, \quad a \geq 2. \quad (4.24)$$

Now repeat the argument from the beginning of this section and assume first that $\Phi_a^a = 0$ for all $a \geq 2$. We obtain

$$\Theta_a^J = 0 \pmod{\varphi}, \quad a \geq 2,$$

and hence df vanishes on the kernel of all θ_1^j and φ_α^β . Since f is an embedding, it follows that the latter kernel equals the full complex tangent space, i.e. $q = 1$. In this case (4.16) implies

$$\Theta_a^J = 0 \pmod{\varphi}, \quad a > 1 = q.$$

Then writing (4.18) and proceeding as before we obtain $\lambda_a^b \varphi_\beta^\alpha = 0$ and hence

$$\Phi_a^b = 0, \quad q = 1. \quad (4.25)$$

(Note that we have assumed $q \geq 2$ excluding this case. However, we shall repeat this procedure when a similar case will occur.)

In the remaining case $q > 1$, our assumption above cannot hold, i.e. $\Phi_a^a \neq 0$ for some a , say $a = 2$. Then (4.14) implies that, after a change of position as before, we may assume that

$$\Phi_2^2 = \sum_{\alpha \geq 2} c_\alpha \varphi_\alpha^\alpha$$

for some $c_\alpha \geq 0$ not all zero. Then (4.1) yields

$$\Theta_2^J \wedge \Theta_J^2 = \sum_{\alpha \geq 2} c_\alpha \theta_\alpha^j \wedge \theta_j^\alpha \pmod{\varphi}. \quad (4.26)$$

Since the proof of Lemma 4.1 can be repeated for Φ_2^2 instead of Φ_1^1 , we conclude that the rank of the left-hand side of (4.26) restricted to the complex tangent space is n . Therefore, in the right-hand side, only one c_α , say c_2 can be different from zero. After a dilation (see Definition 2.3), we may assume

$$\Phi_2^2 = \varphi_2^2$$

and hence

$$\sum_J \Theta_2^J \wedge \overline{\Theta_2^J} = \sum_j \theta_2^j \wedge \overline{\theta_2^j} \pmod{\varphi}. \quad (4.27)$$

We claim that each Θ_2^J is a linear combination of only θ_2^j modulo φ . Indeed, if Θ_2^J were a combination of θ_α^j modulo φ , where some of them enters with a nonzero coefficient λ_α with $\alpha \neq 2$, we would have $\theta_\alpha^j \wedge \overline{\theta_\alpha^j}$ entering with positive coefficient $\geq \lambda_\alpha \overline{\lambda_\alpha}$ in the right-hand side of (4.27), which is impossible, proving our claim. As in the proof of Lemma 4.1 we now write

$$\Theta_2^J = h_J^J \theta_2^j \pmod{\varphi}. \quad (4.28)$$

Since

$$\Phi_2^1 = \lambda_\beta^\alpha \varphi_\alpha^\beta \quad (4.29)$$

for suitable λ_α^β , we obtain

$$\Theta_2^J \wedge \Theta_J^1 = \lambda_\beta^\alpha \theta_\alpha^j \wedge \theta_j^\beta \pmod{\varphi}, \quad (4.30)$$

which in view of (4.28) and Lemma 4.1, yields

$$h_J^k \theta_2^j \wedge \theta_k^1 = \lambda_\beta^\alpha \theta_\alpha^j \wedge \theta_j^\beta \pmod{\varphi}. \quad (4.31)$$

Since the right-hand side contains no terms $\theta_\alpha^j \wedge \theta_k^\beta$ with $j \neq k$, it follows that $h_J^k = 0$ for $j \neq k$ and hence $h_J^j = \lambda_2^1 =: \lambda$ for all j and $\lambda_\beta^\alpha = 0$ for $(\alpha, \beta) \neq (1, 2)$. Then (4.28) implies

$$\Theta_2^j = \lambda \theta_2^j \pmod{\varphi}. \quad (4.32)$$

Finally, substituting (4.32) into (4.27) and identifying coefficients we obtain

$$\lambda \bar{\lambda} \delta_{ij} + \sum_{J>n} h_i^J \overline{h_j^J} = \delta_{ij}.$$

In particular, it follows that the vectors $h_i := (h_i^{n+1}, \dots, h_i^{n'})$ are orthogonal and of the same length. But since we have assumed $n' - n < n$, we must have $h_i = 0$ and therefore $|\lambda| = 1$. Now we perform a change of position as in Definition 2.3 with $W_\alpha^\beta := c_\alpha \delta_{\alpha\beta}$ with $c_\alpha = 1$ for $\alpha \neq 2$ and $c_2 = \lambda$. Then we arrive at the following relations:

$$\Phi_a^a = \varphi_a^a, \quad a = 1, 2, \quad (4.33)$$

$$\Theta_a^J = \theta_a^J \mod \varphi, \quad a = 1, 2. \quad (4.34)$$

Since after the last change of position, we have $\lambda = 1$ in (4.32), we obtain from (4.31) that $\lambda_\beta^\alpha = 0$ unless $\alpha = 2$ and $\beta = 1$, in which case $\lambda_2^1 = 1$. Then substituting into (4.29) yields

$$\Phi_2^1 = \varphi_2^1. \quad (4.35)$$

4.3 Determination of Φ_a^b

Now we repeat again the arguments after the proof of Lemma 4.1, where we replace 1 by 2 and 2 by 3, to arrive at the identities:

$$\Phi_a^2 = 0 \mod \{\varphi_\alpha^\beta : \alpha \geq 3 \text{ or } \beta \geq 3\}, \quad a \geq 3, \quad (4.36)$$

$$\Phi_a^b = 0 \mod \{\varphi_\alpha^\beta : \alpha, \beta \geq 3\}, \quad a, b \geq 3, \quad (4.37)$$

$$\Theta_a^J = 0 \mod \{\theta_\alpha : \alpha \geq 3\}, \varphi, \quad a \geq 3. \quad (4.38)$$

Then continuing following the arguments after (4.24) with the same replacements, we obtain

$$\Phi_3^\alpha = \varphi_3^\alpha, \quad \alpha = 1, 2, 3, \quad (4.39)$$

$$\Theta_3^J = \theta_3^J \mod \varphi. \quad (4.40)$$

Finally, arguing by induction on $b = 4, \dots, q'$, and proceeding by repeating the same arguments with 1 replaced by b and 2 by $b + 1$, we obtain the following lemma.

Lemma 4.2 *For any local CR embedding f from $S_{p,q}$ into $S_{p',q'}$, there is a choice of sections of the bundles $\mathcal{B}_{p,q} \rightarrow S_{p,q}$ and $\mathcal{B}_{p',q'} \rightarrow S_{p',q'}$ such that the pulled back forms satisfy*

$$\Phi_a^b - \varphi_a^b = 0,$$

$$\Theta_a^J - \theta_a^J = 0 \mod \varphi.$$

Remark 4.3 A change of section of $\mathcal{B}_{p,q} \rightarrow S_{p,q}$ (corresponding to a change of frame on $S_{p,q}$) has been used in course of the proof. However, once Lemma 4.2 has been established, one can change the frame on $S_{p,q}$ back to the original one together with the corresponding change of the frame on $S_{p',q'}$ involving only the subframe (Z_a, X_J, Y_b) with $a, b \leq q, J \leq n$, such that the conclusion of the lemma remains valid.

5 Determination of Θ_a^J

In our next analysis we shall use the full structure equations for φ and Φ which in view of Lemma 4.2 take the form

$$d\varphi_a^b = \psi_a^\gamma \wedge \varphi_\gamma^b + \theta_a^j \wedge \theta_j^b + \varphi_a^\gamma \wedge \widehat{\psi}_\gamma^b, \quad (5.1)$$

$$d\varphi_a^b = \Psi_a^\gamma \wedge \varphi_\gamma^b + \Theta_a^J \wedge \Theta_J^b + \varphi_a^\gamma \wedge \widehat{\Psi}_\gamma^b, \quad (5.2)$$

and their difference

$$(\Psi_a^\gamma - \psi_a^\gamma) \wedge \varphi_\gamma^b + \Theta_a^J \wedge \Theta_J^b - \theta_a^j \wedge \theta_j^b + \varphi_a^\gamma \wedge (\widehat{\Psi}_\gamma^b - \widehat{\psi}_\gamma^b) = 0, \quad (5.3)$$

as well as the structure equations for θ and Θ :

$$d\theta_a^J = \psi_a^\beta \wedge \theta_\beta^J + \theta_a^k \wedge \omega_k^J + \varphi_a^\beta \wedge \sigma_\beta^J, \quad (5.4)$$

$$d\Theta_a^J = \Psi_a^b \wedge \Theta_b^J + \Theta_a^K \wedge \Omega_K^J + \varphi_a^\beta \wedge \Sigma_\beta^J. \quad (5.5)$$

Our next goal is to determine Θ_a^J . It will be determined together with components Ψ_a^b and Ω_K^J modulo φ . In view of Lemma 4.2 we can write

$$\Theta_a^J - \theta_a^J = \eta_a^{J\gamma} \varphi_\gamma^\beta, \quad (5.6)$$

for some $\eta_a^{J\gamma}$, and using the symmetry relations (2.8),

$$\Theta_J^a - \theta_J^a = \eta_J^{a\beta} \varphi_\beta^\gamma, \quad (5.7)$$

where

$$\eta_J^{a\beta} := \overline{\eta_a^{J\gamma}}. \quad (5.8)$$

We will show that after a frame change, we may assume that

$$\eta_a^{J\gamma} = 0.$$

Using (5.6) we compute

$$\Theta_a^J \wedge \Theta_J^b - \theta_a^j \wedge \theta_j^b = \eta_a^{j\gamma} \varphi_\gamma^\beta \wedge \theta_j^b - \theta_a^j \wedge \eta_j^{b\beta} \varphi_\beta^\gamma \mod \varphi \wedge \varphi \quad (5.9)$$

and (5.3) becomes

$$\begin{aligned} & (\Psi_a^\gamma - \psi_a^\gamma) \wedge \varphi_\gamma^b + \varphi_a^\gamma \wedge (\widehat{\Psi}_\gamma^b - \widehat{\psi}_\gamma^b) \\ & + \eta_a^{j\beta} \varphi_\gamma^\beta \wedge \theta_j^b - \theta_a^j \wedge \eta_j^{b\beta} \varphi_\beta^\gamma = 0, \mod \varphi \wedge \varphi \end{aligned} \quad (5.10)$$

where $\varphi \wedge \varphi$ stands for the space generated by all possible wedge products $\varphi_\beta^\gamma \wedge \varphi_\delta^\tau$. Together with (5.10) we shall consider the structure equations obtained by differentiating (5.6) and using (4.1), (5.4) and (5.5):

$$\eta_a^J \varphi_\beta^\gamma \theta_\gamma^k \wedge \theta_k^\beta = (\psi_a^\beta - \psi_a^\beta) \wedge \theta_\beta^J + \theta_a^k \wedge (\Omega_k^J - \omega_k^J) \mod \varphi. \quad (5.11)$$

5.1 Determination of Θ_a^J for $a > q$, $J > n$

In case $a > q$ and $J > n$ the right-hand side of (5.11) is zero. Since the forms θ_γ^k and θ_k^β are $(1, 0)$ and $(0, 1)$ respectively and are linearly independent, we conclude

$$\eta_a^J \varphi_\beta^\gamma = 0, \quad a > q, \quad J > n,$$

and hence (5.6) yields

$$\Theta_a^J = 0, \quad a > q, \quad J > n. \quad (5.12)$$

5.2 Determination of Θ_a^j for $a > q$

For $a > q$, $b = \beta \leq q$, (5.10) takes the form

$$0 = \Psi_a^\alpha \wedge \varphi_\alpha^\beta + \eta_a^j \varphi_\delta^\alpha \wedge \theta_j^\beta \mod \varphi \wedge \varphi, \quad a > q. \quad (5.13)$$

Since the forms $\Re(\varphi_\beta^\alpha)$, $\alpha < \beta$, and $\Im(\varphi_\beta^\alpha)$, $\alpha \leq \beta$, are linearly independent over \mathbb{R} , it follows that φ_α^β are linearly independent over \mathbb{C} . Then we can collect the coefficients in front of these forms and apply complex Cartan's Lemma for a fixed β to obtain

$$\Psi_a^\alpha \in \text{span}\{\theta_j^\beta, \varphi\}, \quad a > q.$$

But Ψ_a^α is independent of the choice of β . Therefore, since θ_j^β are linearly independent and we have assumed $q \geq 2$, we conclude that

$$\Psi_a^\alpha = 0 \mod \varphi, \quad a > q, \quad (5.14)$$

and hence (5.13) implies

$$\eta_a^k \varphi_\gamma^\alpha = 0, \quad a > q,$$

which in view of (5.6) yields

$$\Theta_a^j = 0, \quad a > q. \quad (5.15)$$

5.3 Reducing the freedom for Θ_α^J for $J > n$

Next we use (5.11) in case $a = \alpha \leq q$ and $J > n$ that becomes

$$\theta_\alpha^k \wedge \Omega_k^J = \eta_\alpha^J \theta_\gamma^k \wedge \theta_k^\beta \mod \varphi, \quad J > n. \quad (5.16)$$

Since θ_α^k are linearly independent and $(1, 0)$, whereas θ_k^β are $(0, 1)$, the term with $\theta_\gamma^k \wedge \theta_k^\beta$, $\gamma \neq \alpha$, in the right-hand side cannot occur in the left-hand side. Therefore

$$\eta_\alpha^J \theta_\beta^k = 0 \quad \text{if } \gamma \neq \alpha, \quad J > n,$$

and hence (5.6) becomes

$$\Theta_\alpha^J = \eta_\alpha^J \varphi_\alpha^\beta, \quad J > n, \quad (5.17)$$

where

$$\eta_\alpha^J \theta_\beta^k := \eta_\alpha^J \theta_\beta^k,$$

and (5.16) becomes

$$\theta_\alpha^k \wedge \Omega_k^J = \eta_\alpha^J \theta_\alpha^k \wedge \theta_k^\beta \mod \varphi, \quad J > n, \quad (5.18)$$

i.e.

$$\theta_\alpha^k \wedge (\Omega_k^J - \eta_\alpha^J \theta_k^\beta) = 0 \mod \varphi, \quad J > n. \quad (5.19)$$

Then using linear independence of θ_α^k and applying Cartan's Lemma, we obtain

$$\Omega_k^J = \eta_\alpha^J \theta_k^\beta \mod \{\varphi, \theta_\alpha\}, \quad J > n. \quad (5.20)$$

Since Ω_k^J is independent of α and $q \geq 2$ by our assumption, taking $(0, 1)$ parts we obtain

$$\eta_\alpha^J \theta_\beta^k = \eta_\beta^J, \quad J > n, \quad (5.21)$$

for some η_β^J , hence (5.17) implies

$$\Theta_\alpha^J = \eta_\beta^J \varphi_\alpha^\beta, \quad J > n, \quad (5.22)$$

and (5.20) yields

$$\Omega_k^J = \eta_\beta^J \theta_k^\beta \mod \varphi, \quad J > n, \quad (5.23)$$

where we dropped θ_α on the right-hand side, since now both sides are independent of α and since $q \geq 2$ by our assumption.

5.4 Reducing the freedom for Θ_α^j

Here we use the structure equation (5.10) in case $a = \alpha \leq q$ and $b = \beta \leq q$. Then the forms φ_α^γ , $\gamma \neq \beta$, and φ_δ^β , $\delta \neq \alpha$, together with φ_α^β are linearly independent. Therefore identifying the coefficients in front of these forms in (5.10) and using Cartan's Lemma yields

$$\Psi_\alpha^\gamma - \psi_\alpha^\gamma = 0 \pmod{\{\theta_\alpha, \overline{\theta_\beta}, \varphi\}} \quad \text{if } \gamma \neq \alpha, \quad (5.24)$$

$$(\Psi_\alpha^\alpha - \psi_\alpha^\alpha) - (\widehat{\Psi}_\beta^\beta - \widehat{\psi}_\beta^\beta) = 0 \pmod{\{\theta_\alpha, \overline{\theta_\beta}, \varphi\}}. \quad (5.25)$$

Since $(\Psi_\alpha^\gamma - \psi_\alpha^\gamma)$, $\gamma \neq \alpha$, is independent of the choice of β and $q \geq 2$, we conclude

$$\Psi_\alpha^\gamma - \psi_\alpha^\gamma = 0 \pmod{\{\theta_\alpha, \varphi\}} \quad \text{if } \gamma \neq \alpha. \quad (5.26)$$

Substituting now (5.26) into (5.11) for $a = \alpha \leq q$, $J = j \leq n$, and identifying coefficients in front of $\theta_\gamma^k \wedge \theta_k^\beta$ for $\gamma \neq \alpha$, since θ_γ^j are $(1, 0)$ and linearly independent, whereas $\theta_j^\alpha = -\overline{\theta_\alpha^j}$ are $(0, 1)$, we obtain

$$\eta_{\alpha\beta}^{j\gamma} = 0 \quad \text{if } \gamma \neq \alpha.$$

In view of (5.26) we can write

$$\Psi_\alpha^\gamma - \psi_\alpha^\gamma = h_{\alpha k}^\gamma \theta_\alpha^k \pmod{\varphi}, \quad \gamma \neq \alpha, \quad (5.27)$$

for suitable $h_{\alpha k}^\gamma$ and put

$$h_{\alpha k}^\alpha := 0. \quad (5.28)$$

Then (5.11) for $a = \alpha \leq q$ and $J = j \leq n$ becomes

$$\begin{aligned} & \{(\Psi_\alpha^\alpha - \psi_\alpha^\alpha)\delta_k^j - (\Omega_k^j - \omega_k^j)\} \wedge \theta_\alpha^k \\ & + \sum_{\gamma \neq \alpha} (\Psi_\alpha^\gamma - \psi_\alpha^\gamma) \wedge \theta_\gamma^j = \eta_{\alpha\gamma}^j \theta_\alpha^k \wedge \theta_k^\gamma \pmod{\varphi}, \end{aligned}$$

where

$$\eta_{\alpha\beta}^j := \eta_{\alpha\beta}^{j\alpha}.$$

Substituting (5.27) yields

$$\{(\Psi_\alpha^\alpha - \psi_\alpha^\alpha)\delta_k^j - (\Omega_k^j - \omega_k^j + h_{\alpha k}^\gamma \theta_\gamma^j - \eta_{\alpha\gamma}^j \theta_k^\gamma)\} \wedge \theta_\alpha^k = 0 \pmod{\varphi}.$$

Now using Cartan's Lemma we obtain

$$(\Psi_\alpha^\alpha - \psi_\alpha^\alpha)\delta_k^j - (\Omega_k^j - \omega_k^j) = -h_{\alpha k}^\gamma \theta_\gamma^j + \eta_{\alpha\gamma}^j \theta_k^\gamma \pmod{\{\varphi, \theta_\alpha\}}. \quad (5.29)$$

As consequence of (5.25) and (2.8) we also have

$$(\Psi_\alpha^\alpha - \psi_\alpha^\alpha) + (\Psi_{\bar{\beta}}^{\bar{\beta}} - \psi_{\bar{\beta}}^{\bar{\beta}}) = 0 \pmod{\{\theta_\alpha, \overline{\theta_\beta}, \varphi\}}, \quad (5.30)$$

and, in particular,

$$\Re(\Psi_\alpha^\alpha - \psi_\alpha^\alpha) = 0 \pmod{\{\theta_\alpha, \overline{\theta_\alpha}, \varphi\}}. \quad (5.31)$$

Furthermore, since Ω_j^k and ω_j^k are antihermitian in view of (2.8), taking hermitian part with respect to (j, k) of (5.29), using (5.30) and identifying coefficients in front of $\theta_k^\gamma = -\overline{\theta_\gamma^k}$, we obtain

$$\overline{h_\alpha^\gamma} + \eta_\alpha^j{}_\gamma = 0, \quad \text{if } \gamma \neq \alpha. \quad (5.32)$$

Next, since Ω_j^k and ω_j^k are antihermitian, (5.30) implies

$$(\Psi_\alpha^\alpha - \psi_\alpha^\alpha) - (\Omega_j^j - \omega_j^j) = -(\Psi_{\bar{\beta}}^{\bar{\beta}} - \psi_{\bar{\beta}}^{\bar{\beta}}) + (\Omega_{\bar{j}}^{\bar{j}} - \omega_{\bar{j}}^{\bar{j}}) \pmod{\{\varphi, \theta_\alpha, \overline{\theta_\beta}\}}.$$

Hence using (5.29) for $k = j$ and adding its conjugate with α replaced by β , using (5.32) and identifying the coefficients in front of θ_j^γ we obtain

$$\eta_\alpha^j{}_\gamma = \eta_\beta^j{}_\gamma, \quad \gamma \neq \beta,$$

and hence

$$\eta_\alpha^j{}_\gamma = \eta_\gamma^j$$

for suitable η_γ^j . Hence (5.6) implies

$$\Theta_\alpha^j - \theta_\alpha^j = \eta_\beta^j \varphi_\alpha^\beta. \quad (5.33)$$

5.5 Determination of Θ_a^J by a change of frame

Let (B_a^J) be a matrix defined by

$$B_\alpha^J := \eta_\alpha^J, \quad B_a^J := 0, \quad a > q,$$

where η_α^J is defined by (5.21) for $J > n$. Consider the change of frame of $S_{p', q'}$ discussed after Definition 2.3, given by

$$\widetilde{Z}_a = Z_a, \quad \widetilde{X}_J = X_J + C_J^b Z_b, \quad \widetilde{Y}_a = Y_a + A_a^b Z_b + B_a^J X_J$$

such that

$$C_J^a := -B_J^a$$

and A_a^b satisfies

$$(A_a^b + \overline{A_b^a}) + \sum_J B_a^J \overline{B_b^J} = 0.$$

Since the sum here is hermitian, one can always choose A_a^b with this property. Then $\Phi_a^b = \varphi_a^b$ remain the same while Θ_a^J change to

$$\Theta_a^J - \varphi_a^b B_b^J.$$

Therefore (5.33) becomes

$$\Theta_a^J = \theta_a^J, \quad (5.34)$$

which is equivalent to $\eta_\beta^J = 0$ and hence $\eta_\alpha^j{}_\gamma = 0$ and $h_\alpha^\gamma{}_j = 0$ in view of (5.28) and (5.32). Therefore (5.29) implies

$$(\Psi_\alpha^\alpha - \psi_\alpha^\alpha) \delta_k^j - (\Omega_k^j - \omega_k^j) = 0 \mod \{\varphi, \theta_\alpha\}, \quad (5.35)$$

and (5.27) together with (5.14) implies

$$\Psi_a^\gamma - \psi_a^\gamma = 0 \mod \varphi, \quad \text{if } \gamma \neq a, \quad (5.36)$$

and, since the left-hand side of (5.35) is independent of α for $j \neq k$, together with (5.23) we obtain

$$\Omega_k^J - \omega_k^J = 0 \mod \varphi, \quad \text{if } J \neq k. \quad (5.37)$$

6 Determination of Ψ_a^β and Ω_k^J

Next, we use (5.34) in the structure equations for $d\theta_a^J = d\Theta_a^J$, which yield

$$(\Psi_a^\beta - \psi_a^\beta) \wedge \theta_\beta^J + \theta_a^k \wedge (\Omega_k^J - \omega_k^J) + \varphi_a^\beta \wedge (\Sigma_\beta^J - \sigma_\beta^J) = 0. \quad (6.1)$$

6.1 Determination of Ψ_a^β for $a > q$

In case $a > q$, $J = j \leq n$, (6.1) takes the form

$$\Psi_a^\beta \wedge \theta_\beta^j = 0, \quad a > q. \quad (6.2)$$

Together with (5.36), this yields

$$\Psi_a^\beta = 0, \quad a > q. \quad (6.3)$$

6.2 Determination of Ω_k^J and Σ_β^J for $J > n$

Next use (6.1) for $a = \alpha \leq q$ and $J > n$ to obtain

$$\theta_\alpha^k \wedge \Omega_k^J + \varphi_\alpha^\beta \wedge \Sigma_\beta^J = 0, \quad J > n.$$

By Cartan's Lemma,

$$\Omega_k^J = 0 \pmod{\{\theta_\alpha, \varphi_\alpha\}}, \quad \Sigma_\beta^J = 0 \pmod{\{\theta_\alpha, \varphi_\alpha\}}, \quad J > n, \quad (6.4)$$

where φ_α is the ideal generated by φ_α^β for α fixed. Since Ω_k^J and Σ_β^J are independent of α and $q \geq 2$, we conclude

$$\Omega_k^J = \Sigma_\beta^J = 0, \quad J > n. \quad (6.5)$$

We summarize the obtained alignment of the connection forms:

Proposition 6.1 *For any local CR embedding f from $S_{p,q}$ into $S_{p',q'}$, there is a choice of sections of the frame bundles $\mathcal{B}_{p,q} \rightarrow S_{p,q}$ and $\mathcal{B}_{p',q'} \rightarrow S_{p',q'}$ such that*

$$\Phi_a^b = \varphi_a^b, \quad \Theta_a^J = \theta_a^J, \quad (6.6)$$

$$\Psi_a^\beta = 0, \quad \Omega_j^K = 0, \quad \Sigma_\alpha^K = 0, \quad a > q, \quad K > n. \quad (6.7)$$

Remark 6.2 Similarly to Remark 4.3, we can restrict to changing only the section of the second bundle $\mathcal{B}_{p',q'} \rightarrow S_{p',q'}$.

7 Embeddability in a plane of suitable dimension

Proposition 7.1 *Under the assumptions of Theorem 1.1, there exist a $(p+q)$ -dimensional subspace V_1 and a $(q' - q)$ -dimensional subspace V_2 in $\mathbb{C}^{p'+q'}$ with $V_1 \cap V_2 = 0$ and such that $\langle \cdot, \cdot \rangle$ nondegenerate of signature (p, q) when restricted to V_1 and null when restricted to V_2 such that $f(S_{p,q}) \subset \text{Gr}(V_1, q) \oplus V_2$.*

Proof Denote by $M \subset S_{p,q}$ the open subset where f is defined. Let Z, X, Y be constant vector fields of $\mathbb{C}^{p'+q'}$ forming a $S_{p',q'}$ -frame adapted to M at a fixed reference point in M . Let

$$\widetilde{Z}_a = \lambda_a^b Z_b + \eta_a^K X_K + \zeta_a^b Y_b, \quad (7.1)$$

$$\widetilde{X}_J = \lambda_J^b Z_b + \eta_J^K X_K + \zeta_J^b Y_b, \quad (7.2)$$

$$\widetilde{Y}_a = \tilde{\lambda}_a^b Z_b + \tilde{\eta}_a^K X_K + \tilde{\zeta}_a^b Y_b \quad (7.3)$$

be an adapted $S_{p',q'}$ -frame along M . Write

$$A = \begin{pmatrix} \lambda_a^b & \eta_a^K & \zeta_a^b \\ \lambda_J^b & \eta_J^K & \zeta_J^b \\ \tilde{\lambda}_a^b & \tilde{\eta}_a^K & \tilde{\zeta}_a^b \end{pmatrix}, \quad (7.4)$$

so that (7.1)–(7.3) take the form

$$\begin{pmatrix} \tilde{Z} \\ \tilde{X} \\ \tilde{Y} \end{pmatrix} = A \begin{pmatrix} Z \\ X \\ Y \end{pmatrix}. \quad (7.5)$$

Since Z, X, Y form an adapted frame at a reference point of M , we may assume that

$$A = I_{p'+q'} \quad (7.6)$$

at the reference point. Since Z, X, Y are constant vector fields, i.e., $dZ = dX = dY = 0$, differentiating (7.5) and using (2.5) we obtain

$$dA = \Pi A, \quad (7.7)$$

where Π is the connection matrix of $S_{p',q'}$, i.e.

$$dA = \begin{pmatrix} \Psi_a^b & \Theta_a^J & \Phi_a^b \\ \Sigma_K^b & \Omega_K^J & \Theta_K^b \\ \Xi_a^b & \Sigma_a^J & \hat{\Psi}_a^b \end{pmatrix} A. \quad (7.8)$$

Next, it follows from Proposition 6.1 that

$$d\tilde{Z}_a = \sum_{b>q} \Psi_a^b \tilde{Z}_b, \quad a > q, \quad (7.9)$$

in particular, the span of \tilde{Z}_a , $a > q$, is independent of the point in M . Hence together with (7.1) and (7.6), we conclude

$$\eta_a^K = \zeta_a^b = 0, \quad a > q. \quad (7.10)$$

Furthermore, (7.8) implies

$$\begin{pmatrix} d\eta_a^K \\ d\eta_J^K \\ d\tilde{\eta}_a^K \end{pmatrix} = \begin{pmatrix} \Psi_a^b & \Theta_a^L & \Phi_a^b \\ \Sigma_J^b & \Omega_J^L & \Theta_J^b \\ \Xi_a^b & \Sigma_a^L & \hat{\Psi}_a^b \end{pmatrix} \begin{pmatrix} \eta_b^K \\ \eta_L^K \\ \tilde{\eta}_b^K \end{pmatrix}. \quad (7.11)$$

In particular, restricting to $a = \alpha \leq q$ and $J = j \leq n$ and using Proposition 6.1 (together with the symmetry relations analogous to (2.8)) we obtain

$$\begin{pmatrix} d\eta_\alpha^K \\ d\eta_j^K \\ d\tilde{\eta}_\alpha^K \end{pmatrix} = \begin{pmatrix} \Psi_\alpha^b & \theta_\alpha^L & \varphi_\alpha^b \\ \Sigma_j^b & \Omega_j^L & \theta_j^b \\ \Xi_\alpha^b & \Sigma_\alpha^L & \widehat{\Psi}_\alpha^b \end{pmatrix} \begin{pmatrix} \eta_b^K \\ \eta_L^K \\ \tilde{\eta}_b^K \end{pmatrix}. \quad (7.12)$$

Now with (7.10) and Proposition 6.1 taken into account, (7.12) becomes

$$\begin{pmatrix} d\eta_\alpha^K \\ d\eta_j^K \\ d\tilde{\eta}_\alpha^K \end{pmatrix} = \begin{pmatrix} \Psi_\alpha^\beta & \theta_\alpha^l & \varphi_\alpha^\beta \\ \Sigma_j^\beta & \Omega_j^l & \theta_j^\beta \\ \Xi_\alpha^\beta & \Sigma_\alpha^l & \widehat{\Psi}_\alpha^\beta \end{pmatrix} \begin{pmatrix} \eta_\beta^K \\ \eta_l^K \\ \tilde{\eta}_\beta^K \end{pmatrix}. \quad (7.13)$$

Repeating the above argument for ζ instead of η , we obtain

$$\begin{pmatrix} d\zeta_\alpha^b \\ d\zeta_j^b \\ d\tilde{\zeta}_\alpha^b \end{pmatrix} = \begin{pmatrix} \Psi_\alpha^\beta & \theta_\alpha^l & \varphi_\alpha^\beta \\ \Sigma_j^\beta & \Omega_j^l & \theta_j^\beta \\ \Xi_\alpha^\beta & \Sigma_\alpha^l & \widehat{\Psi}_\alpha^\beta \end{pmatrix} \begin{pmatrix} \zeta_\beta^b \\ \zeta_l^b \\ \tilde{\zeta}_\beta^b \end{pmatrix}. \quad (7.14)$$

Thus each of the vector valued functions $\eta^K := (\eta_\alpha^K, \eta_j^K, \tilde{\eta}_\alpha^K)$ for a fixed K and $\zeta^b := (\zeta_\alpha^b, \zeta_j^b, \tilde{\zeta}_\alpha^b)$ for a fixed b satisfies a complete system of linear first order differential equations. Then by the initial condition (7.6) and the uniqueness of solutions, we conclude, in particular, that

$$\eta^K = \zeta^b = 0, \quad K > n, \quad b > q. \quad (7.15)$$

Hence (7.5) implies

$$\tilde{Z}_\alpha = \lambda_\alpha^b Z_b + \eta_\alpha^k X_k + \zeta_\alpha^\beta Y_\beta. \quad (7.16)$$

Now setting

$$\widehat{Z}_\alpha := \tilde{Z}_\alpha - \sum_{b>q} \lambda_\alpha^b Z_b, \quad (7.17)$$

we still have

$$\text{span}\{\widehat{Z}_\alpha, \tilde{Z}_{q+1}, \dots, \tilde{Z}_{q'}\} = \text{span}\{\tilde{Z}_\alpha\}, \quad (7.18)$$

whereas (7.16) becomes

$$\widehat{Z}_\alpha = \lambda_\alpha^\beta Z_\beta + \eta_\alpha^k X_k + \zeta_\alpha^\beta Y_\beta, \quad (7.19)$$

implying

$$\text{span}\{\widehat{Z}_\alpha\} \subset \text{span}\{Z_1, \dots, Z_q, X_1, \dots, X_n, Y_1, \dots, Y_q\}.$$

Then together with (7.9) we conclude that

$$\begin{aligned} f(M) &= \text{span}\{\widetilde{Z}_\alpha\} = \text{span}\{\widehat{Z}_\alpha\} \oplus \text{span}\{\widetilde{Z}_{q+1}, \dots, \widetilde{Z}_{q'}\} \\ &= \text{span}\{\widehat{Z}_\alpha\} \oplus \text{span}\{Z_{q+1}, \dots, Z_{q'}\} \subset Gr(V_1, q) \oplus V_2, \end{aligned}$$

where

$$\begin{aligned} V_1 &= \text{span}\{Z_1, \dots, Z_q, X_1, \dots, X_n, Y_1, \dots, Y_q\}, \\ V_2 &= \text{span}\{Z_{q+1}, \dots, Z_{q'}\}. \end{aligned} \tag{7.20}$$

□

8 Rigidity of CR embeddings from $S_{p,q}$ to $S_{p',q'}$

As consequence of Proposition 7.1, we conclude that, after a linear change of coordinates, $f(M)$ locally coincides with $S_{p,q}$ linearly embedded into $S_{p',q'}$. Identifying $M = S_{p,q}$ with its image, f becomes a local CR-automorphism of M . Then by a theorem of Kaup-Zaitsev [26, Theorem 4.5], f is a restriction of a global CR-automorphism of $S_{p,q}$. Furthermore, by [24, Theorem 8.5] (see also [25]), f extends to a biholomorphic automorphism of the bounded symmetric domain and the rigidity follows.

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