Exercise 1
Show that the following functionals on $C[0,1]$ are bounded linear functionals and find their norms:

(i) \( \varphi(f) = f(1) - f(0) \);

(ii) \( \varphi(f) = 2 \int_0^{1/2} f(t) \, dt - \int_{1/2}^1 f(t) \, dt \).

Exercise 2
Let \( A: V \to W \) be a bounded linear operator between normed spaces and let \( C \geq 0 \) be the smallest constant such that \( \|Ax\| \leq C\|x\| \) for all \( x \in V \). Show that \( C = \|A\| \).

Exercise 3
Let \( A_s: f(t) \mapsto f(t + s) \) be the family of shift operators on \( L^p(\mathbb{R}) \). Show that each \( A_s \) is a bounded linear operator and find its norm.

Exercise 4
Let \( V \) be a normed space, \( V_0 \subset V \) a vector subspace. Consider the quotient space \( V/V_0 \), i.e. the vector space of all cosets \( [x] = x + V_0, \ x \in V \). Define

\[ \|[x]\| := \inf\{\|y\| : y \in [x]\} \]

(i) Show that \( \|[x]\| \) is a norm on \( V/V_0 \) if and only if the subspace \( V_0 \) is closed in \( V \).

(ii) If \( V \) is Banach and \( V_0 \) is closed, show that \( V/V_0 \) with this norm is also Banach.

Exercise 5
Let \( V \) be the space of all sequences \( x = (x_n) \) with \( \|x\| := \sum_n 2^n |x_n| < \infty \). Give a description for the dual space \( V^* \) and its norm.
**Exercise 6**

Consider the differential operator $Af(x) = f'(x)$ from the subspace $C^1[a, b]$ of continuously differentiable functions into $C[a, b]$. Define

$$\|f\|_1 := \max_{a \leq x \leq b} |f(x)| + \max_{a \leq x \leq b} |f'(x)|.$$ 

Show:

(i) $\|f\|_1$ is a norm on $C^1[a, b]$;

(ii) $A: C^1[a, b] \rightarrow C[a, b]$ is a bounded linear operator with respect to this norm.

**Exercise 7**

Prove that every closed subspace of a reflexive space is itself reflexive.

**Exercise 8**

Give an example of a bounded linear operator on a Banach space with infinitely many distinct eigenvalues.