Course 321 2006-07

Sheet 4

Due: after the lecture after the lecture next Thursday

Exercise 1

Show that the following functionals on $C[0,1]$ are bounded linear functionals and find their norms:

(i) $\varphi(f) = f(1) - f(0)$;

(ii) $\varphi(f) = 2\int_0^{1/2} f(t)dt - \int_{1/2}^1 f(t)dt$.

Exercise 2

Let $A\colon V \to W$ be a bounded linear operator between normed spaces and let $C \geq 0$ be the smallest constant such that $\|Ax\| \leq C\|x\|$ for all $x \in V$. Show that $C = \|A\|$.

Exercise 3

Let $A_s\colon f(t) \mapsto f(t+s)$ be the family of shift operators on $L^p(\mathbb{R})$. Show that each A_s is a bounded linear operator and find its norm.

Exercise 4

Let V be a normed space, $V_0 \subset V$ a vector subspace. Consider the quotient space V/V_0, i.e. the vector space of all cosets $[x] = x + V_0$, $x \in V$. Define

$$\|[x]\| := \inf\{\|y\| : y \in [x]\}.$$

(i) Show that $\|[x]\|$ is a norm on V/V_0 if and only if the subspace V_0 is closed in V.

(ii) If V is Banach and V_0 is closed, show that V/V_0 with this norm is also Banach.

Exercise 5

Let V be the space of all sequences $x = (x_n)$ with $\|x\| := \sum_n 2^n |x_n| < \infty$. Give a description for the dual space V^* and its norm.

Exercise 6

Consider the differential operator $Af(x) = f'(x)$ from the subspace $C^1[a,b]$ of continuously differentiable functions into $C[a,b]$. Define

$$\|f\|_1 := \max_{a \le x \le b} |f(x)| + \max_{a \le x \le b} |f'(x)|.$$

Show:

(i) $\|f\|_1$ is a norm on $C^1[a,b]$;

(ii) $A\colon C^1[a,b] \to C[a,b]$ is a bounded linear operator with respect to this norm.

Exercise 7

Prove that every closed subspace of a reflexive space is itself reflexive.

Exercise 8

Give an example of a bounded linear operator on a Banach space with infinitely many distinct eigenvalues.