Course 321 2006-07

S h e e t 3

Due: after the lecture beginning next Term

Exercise 1

(i) Give an example of a sequence of functions in $L^1[0,1]$ which converge to 0 pointwise but not in $L^1[0,1]$.

(ii) Give an example of a sequence of functions in $L^1(\mathbb{R})$ which converge to 0 uniformly but not in $L^1(\mathbb{R})$.

(iii) Is there an example in (ii) with $[0,1]$ instead of $\mathbb{R}$?

Exercise 2

Give an example of a nested sequence of closed subsets F_n in $\mathbb{R}$ with $\cap_n F_n = \emptyset$.

Exercise 3

Give an example of a topological space which is first countable but not second countable.

Exercise 4

Let τ be the collection of subsets of $\mathbb{R}$ consisting of the empty set and all complements of countable subsets (countable sets include finite ones).

(i) Show that τ is a topology on $\mathbb{R}$. Is it Hausdorff?

(ii) Show that $(\mathbb{R}, \tau)$ is neither first nor second countable.

(iii) Show that the only convergent sequences are stationary, i.e. the terms are the same starting from some index.

(iv) Show that the closure of $A = [0,1]$ with respect to τ contains points x for which there is no sequence in A converging to x.

(v) Give an example of a discontinuous function f on $(\mathbb{R}, \tau)$ such that $f(x_n) \to f(x)$ whenever x_n is a sequence with $x_n \to x$ as $n \to \infty$.

Exercise 5

(i) Let (X, d) be a metric space and define $\widetilde{d}(x,y) := \frac{d(x,y)}{1+d(x,y)}$. Show that $\widetilde{d}$ is a metric. (Hint. Use the monotonicity of $\varphi(t) = \frac{t}{1+t}$ to show that $a \le b + c$ implies $\varphi(a) \le \varphi(b) + \varphi(c)$.)

(ii) Consider $\mathbb{R}$ with the standard metric $d(x,y) = |x-y|$ and define $\widetilde{d}$ as above. Is $(\mathbb{R}, \widetilde{d})$ complete?

(iii) For a Fréchet space equipped with seminorms $\|x\|_n$, show that

$$d(x,y) := \sum_n \frac{1}{2^n} \frac{\|x-y\|_n}{1+\|x-y\|_n}$$

defines a metric.

Exercise 6

Let $C(\mathbb{R})$ denote the space of all continuous functions on $\mathbb{R}$ and for $f \in C(\mathbb{R})$ define

$$\|f\|_n := \sup_{-n \le x \le n} |f(x)|.$$

(i) Show that each $\|\cdot\|_n$ is a seminorm but not a norm.

(ii) Show that $C(\mathbb{R})$ equipped with seminorms $\|\cdot\|_n$, $n \in \mathbb{N}$, is a Fréchet space.

Exercise 7

Let $C((a,b))$ be the space of all continuous functions on an open interval (a,b). Consider a family of closed subintervals $[a_n, b_n] \subset (a,b)$, $n \in \mathbb{N}$, such that $\bigcup_n [a_n, b_n] = (a,b)$ and $[a_n, b_n] \subset [a_{n+1}, b_{n+1}]$ and define

$$\|f\|_n := \sup_{x \in [a_n, b_n]} |f(x)|.$$

(i) Show that $C((a,b))$ equipped with seminorms $\|\cdot\|_n$, $n \in \mathbb{N}$, is a Fréchet space.

(ii) Show that the topology on $C((a,b))$ defined by these seminorms is independent of the choice of the intervals $[a_n, b_n]$.

Exercise 8

Let $C^\infty[a,b]$ be the space of all infinitely differentiable real functions on $[a,b]$. For every $f \in C^\infty[a,b]$ and $n \ge 0$, define $\|f\|_n := \sup_{x \in [a,b]} |f^{(n)}(x)|$, where $f^{(n)}$ is the nth derivative. Show that $\|f\|_n$ are seminorms equipping $C^\infty[a,b]$ with the structure of a Fréchet space.

Exercise 9

A set A in a topological vector space (V, τ) is called bounded if, for every neighborhood U of 0, there exists a number $\lambda > 0$ such that $\lambda U \supset A$. Here $\lambda U := \{\lambda x : x \in U\}$. The space (V, τ) is called locally bounded if 0 has a bounded neighborhood.

(i) Show that any normed space is locally bounded.

(ii) Prove that the spaces in Exercises 7 and 8 are not normable, i.e. they do not admit any norm defining the underlined topology. (Hint. Show that they are not locally bounded.)