Exercise 1
A function \( f: \mathbb{R} \to \mathbb{R} \) is said to be convex if
\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)
\]
for all \( x, y \in \mathbb{R} \) and \( \lambda \in [0, 1] \). Prove that any convex function is continuous.

Exercise 2
Prove that if \( B, C \) are bounded subsets of a metric space \( M \) with \( B \cap C \neq \emptyset \), then
diam\((B \cup C) \leq \text{diam}B + \text{diam}C \).

Exercise 3
Prove that a subset of a metric space \( M \) is open if and only if it is a union of open balls. Is the same true for countable unions in case (a) \( M = \mathbb{R}^n \); (b) \( M \) is an arbitrary metric space? Justify your answer by providing a proof or a counterexample if necessary.

Exercise 4
Consider \( \mathbb{R}^2 \) with two different metrics: \( d_2 \) being the standard one and
\[
d_1((x_1, y_1), (x_2, y_2)) := |x_1 - x_2| + |y_1 - y_2|.
\]
Show that a function \( f: \mathbb{R}^2 \to \mathbb{R} \) is continuous with respect to \( d_1 \) if and only if it is continuous with respect to \( d_2 \). Can the target space \( \mathbb{R} \) be replaced by an arbitrary metric space?

Exercise 5
Let \( M := C[a, b] \) be the metric space of all continuous functions on the interval \([a, b]\) with the metric
\[
d(f, g) := \sup_{x \in [a, b]} |f(x) - g(x)|
\]
for \( f, g \in M \). Fix two functions \( f, g \in M \) with \( f(x) < g(x) \) for all \( x \) and consider the set \( U := \{ h \in M : f(x) < h(x) < g(x) \} \). Is \( U \) open in \( M \)? Is it a ball? If not always, what are the (necessary and sufficient) conditions on \( f \) and \( g \) in order that \( U \) be a ball?