Course 212 2004-05

Sheet 5

Due: after the lecture beginning of the next term

Exercise 1

A function $f: \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y)$$

for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$. Prove that any convex function is continuous.

Exercise 2

Prove that if B, C are bounded subsets of a metric space M with $B \cap C \neq \emptyset$, then $\operatorname{diam}(B \cup C) \le \operatorname{diam} B + \operatorname{diam} C$.

Exercise 3

Prove that a subset of a metric space M is open if and only if it is a union of open balls. Is the same true for countable unions in case (a) $M = \mathbb{R}^n$; (b) M is an arbitrary metric space? Justify your answer by providing a proof or a counterexample if necessary.

Exercise 4

Consider $\mathbb{R}^2$ with two different metrics: d_2 being the standard one and

$$d_1((x_1, y_1), (x_2, y_2)) := |x_1 - x_2| + |y_1 - y_2|.$$

Show that a function $f: \mathbb{R}^2 \to \mathbb{R}$ is continuous with respect to d_1 if and only if it is continuous with respect to d_2. Can the target space $\mathbb{R}$ be replaced by an aritrary metric space?

Exercise 5

Let $M := C[a,b]$ be the metric space of all continuous functions on the interval $[a,b]$ with the metric

$$d(f,g) := \sup_{x \in [a,b]} |f(x) - g(x)|$$

for $f, g \in M$. Fix two functions $f, g \in M$ with $f(x) < g(x)$ for all x and consider the set $U := \{h \in M : f(x) < h(x) < g(x)\}$. Is U open in M? Is it a ball? If not always, what are the (necessary and sufficient) conditions on f and g in order that U be a ball?