New tools and conditions for global regularity of the $\bar{\partial}$-Neumann operator

Part 3 - Tower multitype and contact type

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Notation: local defining functions

1. *smooth* means always C^∞;
2. $S \subset \mathbb{C}^{n+1}$, $n \geq 1$, (or $S \subset \mathbb{C}^n$, $n \geq 2$) is a smooth real hypersurface;
3. a *local defining function* r of S in a neighborhood U of $p \in S$ is any smooth real function with

$$S \cap U = \{r = 0\}$$

   and $dr \neq 0$ at every point of U;
4. any two local defining functions r_1, r_2 in U differ by a *nonzero smooth function factor*:

$$r_2(x) = a(x) r_1(x), \quad a(x) \neq 0.$$

Notation: tangent bundles

1. TS is the *real tangent bundle*;
2. $\mathbb{C}TS = \mathbb{C} \otimes_{\mathbb{R}} TS$ is the *complexified tangent bundle*;
3. $H^{10}S = \{X \in \mathbb{C}TS : \partial r(X) = 0\}$ is the $(1,0)$ bundle;
4. $H^{01}S = \{X \in \mathbb{C}TS : \bar{\partial} r(X) = 0\}$ is the $(0,1)$ bundle;
5. $HS = \mathrm{Re} H^{10}S = \mathrm{Re} H^{01}S \subset TS$ is the *complex tangent bundle*;
6. We have the standard relations:

$$H^{01}S = \overline{H^{10}S}, \quad \mathbb{C}HS = H^{10}S \oplus \overline{H^{10}S}.$$

Motivation: from finiteness conditions to global regularity

$$
\begin{array}{ccc}
\text{finiteness conditions} & \Longrightarrow & \text{generalized stratifications} \\
 & & \Downarrow \\
\text{condition } R & & \text{property } (P_q) \\
\Uparrow & & \Downarrow \\
\text{global regularity} & \Longleftarrow & \text{compactness}
\end{array}
$$

Shown in Lectures 1 & 2:

$$
\begin{array}{ccc}
\text{finite Levi or commutator type} & \Longrightarrow & \text{finite tower multitype} \\
 & & \Downarrow \\
 & & \text{generalized stratifications}
\end{array}
$$

Our next goal will be to extend the first implication to another finiteness condition based on *order of contact* and the *contact type*.

Order of contact

Definition

The *order of contact* or *contact order* $o(\gamma, S)$ at $p \in S$ between S and a formal holomorphic immersion (regular formal curve) $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^n, p)$ is the vanishing order of $j_p^\infty r \circ \gamma$, where $j_p^\infty r$ is the Taylor series of r at p.

Our next goal is to show that *finite contact order* for all formal holomorphic immersions from $(\mathbb{C}, 0)$ implies finite tower multitype:

Theorem (main theorem for finite contact type in case $q = 1$)

For a pseudoconvex smooth hypersurface $S \subset \mathbb{C}^n$, assume $o(\gamma, S) < \infty$ for any formal holomorphic immersion $\gamma\colon (\mathbb{C}, 0) \to (\mathbb{C}^n, p)$. Then S is of finite tower multitype *at p.*

This can be combined with previously shown implications to conclude compactness and global regularity of the $\bar\partial$-Neumann operator.
The assumption of the theorem always holds when S is of *finite regular type* at p, hence, in particular, when it is of *finite D'Angelo (singular) type*.

Recall from Lecture 1: Dual forms and special subbundles

The *θ-dual form* of an (ordered) list of complex vector fields

$$L^t, \dots, L^1 \in \Gamma(H^{10}S) \cup \Gamma(\overline{H^{10}S}), \quad t \geq 1,$$

is the complex 1-form $\omega_{L^t,\dots,L^1;\theta}$ on $H^{10}S$ defined for $L \in \Gamma(H^{10}S)$, $p \in S$, by

$$\begin{cases} \omega_{L^1;\theta}(L_p) := \theta([L, L^1])(p) & t = 1 \\ \omega_{L^t,\dots,L^1;\theta}(L_p) := L\mathrm{Re}(L^t \cdots L^3\theta([L^2, L^1]))(p), & t \geq 2 \end{cases}.$$

Definition (special subbundle)

A complex subbundle $E \subset H^{10}S$ is called special if it can be defined by

$$E = \{\xi \in H^{10}S : \omega_1(\xi) = \ldots = \omega_l(\xi) = 0\}, \quad \omega_1 \wedge \cdots \wedge \omega_l \neq 0 \text{ on } (H^{10}S)^l,$$

where each ω_j, $j = 1, \dots, l$, is the *θ-dual 1-form* $\omega_j = \omega_{L_j^{t_j},\dots,L_j^1}$ for some $t_j \geq 1$ and vector fields $L_j^{t_j}, \dots, L_j^1 \in \Gamma(H^{10}S) \cup \Gamma(\overline{H^{10}S})$.

Reduction to finite Levi type

Recall from Lecture 1:

Definition (Levi type of a subbundle)

The Levi type $c(E,p) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ at $p \in S$ of a subbundle $E \subset H^{10}S$ is

$$\min\{t \geq 2 : \exists L^t, \ldots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E}), L^m \cdots L^3 \partial r([L^2, L^1])(p) \neq 0\},$$

where r is a local defining function of S.

Theorem (assuming finite Levi type only for special subbundles)

Assume Levi type $c(E,p) < \infty$ for any special subbundle $E \subset H^{10}S$ of rank ≥ 1. Then S is of finite tower multitype at p.

Thus, to show finite tower multitype, it suffices to prove the *finiteness of Levi types of special subbundles*. Hence we shall assume there is a special subbundle $E \subset H^{10}S$ of *infinite Levi type* and reach a contradiction constructing a formal immersion γ with $o(\gamma, S) = \infty$.

Recall: real and complex formal orbits of subbundles

Our main tool for constructing formal immersions are formal orbits.

Definition (recall from Lecture 2)

1. Let $\mathcal{L}$ be a set of smooth vector fields in a neighborhood of p in $\mathbb{C}^n$, and $\mathcal{R} \subset \mathbb{C}[[z-p, \overline{z-p}]]$ a subring of formal power series. The formal orbit $O = \mathcal{O}^{\mathcal{R}}_{\mathcal{L}}(p)$ is the formal variety given by the *formal power series ideal*

$$I(O) = \{f \in \mathcal{R} : L^t \cdots L^1 f(p) = 0,\ L^t, \dots, L^1 \in \mathcal{L},\ t \geq 0\}.$$

2. The *real formal orbit* (resp. *complex formal orbit*) of a subbundle $E \subset H^{10}S$ is

$$\mathcal{O}^{\mathbb{R}}_E := \mathcal{O}^{\mathcal{R}_{\mathbb{R}}}_{\mathcal{L}(E)} \text{ (resp. } \mathcal{O}^{\mathbb{C}}_E := \mathcal{O}^{\mathcal{R}_{\mathbb{C}}}_{\mathcal{L}(E)}),$$

where $\mathcal{R}_{\mathbb{R}} := \{f \in \mathbb{C}[[z-p, \overline{z-p}]] : \overline{f} = f\}$ and $\mathcal{R}_{\mathbb{C}} := \mathbb{C}[[z-p]]$ and

$$\mathcal{L} = \mathcal{L}(E) = \{L : L|_S \in \Gamma(E) \cup \Gamma(\overline{E})\}.$$

Formal submanifolds and manifold ideals

Recall from Lecture 2: *real* formal orbits of smooth subbundles are always so-called *formal submanifolds*:

Definition (Recall from Lecture 2)

1. A manifold ideal I in a ring $\mathcal{R} \subset \mathbb{C}[[z-p, \overline{z-p}]]$ is one generated by k elements $F_1, \ldots, F_k \in I$ with linearly independent differentials at p.
2. A *real (resp. complex) formal submanifold* O at p in $\mathbb{C}^n$ is one defined by a *manifold ideal* $I = I(O)$ in $\mathcal{R}_{\mathbb{R}}$ (resp. $\mathcal{R}_{\mathbb{C}}$).

We need to construct formal holomorphic immersions, obtained using *formal parametrizations* of *formal complex submanifolds*. Unlike in the real category, the latter are not always complex formal submanifolds, i.e. may be singular, but we still prove in our case:

Theorem

Let $E \subset H^{10}S$ be a special subbundle of infinite Levi type at p. Then the complex formal orbit *$\mathcal{O}_E^{\mathbb{C}}(p)$ is a* complex formal submanifold.

Intrinsic complexification

The complex formal orbit can be viewed as the *intrinsic complexification* of the real formal orbit.

Definition

The *intrinsic complexification* of

1. a real-analytic subset $A \subset \mathbb{C}^n$ at $p \in A$ is the minimum germ of a complex-analytic subset containing A.
2. a real formal variety given by an ideal $I \subset \mathcal{R}_{\mathbb{R}}$ is the complex formal variety given by the ideal $\mathbb{C}I \cap \mathcal{R}_{\mathbb{C}}$

Intrinsic complexification is *regular* for CR submanifolds:

Theorem

The intrinsic complexification of a real-analytic CR submanifold *is a complex submanifold.*

We need a formal variant of this result, for which we need to define *formal CR submanifolds*.

Formal CR submanifolds

A real submanifold $O \subset \mathbb{C}^n$ is CR whenever $H^{10}_x O$, $x \in O$, has constant dimension. This definition cannot be restated for formal submanifolds O given by formal power series ideals $I \subset \mathcal{R}_{\mathbb{R}}$, since *$H^{10}_x O$ is not defined for $x \neq p$*. Instead, one defines $(1,0)$ formal vector fields:

$$\mathcal{D}^{10}_O = \left\{ L = \sum a_j(z,\bar{z})\partial_{z_j} : L(\mathbb{C}I) \subset \mathbb{C}I \right\},$$

and following Della Salla-Juhlin-Lamel, defines:

Definition

A real *formal submanifold* $O \subset \mathbb{C}^n$ at p is said to be *CR* if

$$H^{10}_p O \subset \mathcal{D}^{10}_O(p),$$

i.e. if vector fields from $\mathcal{D}^{10}_O$ span $H^{10}_p O$.

The above definition avoids the mentioned problem of evaluating $H^{10}_x O$ for $x \neq p$ and is equivalent to the CR condition for submanifolds.

Formal Huang-Yin condition

In their detailed study of different notions of types for subbundles $E \subset H^{10}S$, Huang and Yin formulated an important condition on the Nagano leaf of a weighted truncation of E to be CR. We shall need an analogous condition on the real formal orbit that we call the formal Huang-Yin condition:

Definition

We say that a smooth complex subbundle $E \subset H^{10}S$ satisfies the formal Huang-Yin condition at $p \in S$ if the real formal orbit $\mathcal{O}^{\mathbb{R}}_E(p)$ is CR.

We show that condition always holds in our case:

Theorem

Let S be pseudoconvex and $E \subset H^{10}S$ be a special subbundle of infinite Levi type. Then E satisfies the formal Huang-Yin condition, i.e. the real formal orbit $O = \mathcal{O}^{\mathbb{R}}_E$ is CR.

Since E is special, it is defined by

$$E = \{\omega_{L_1;\theta} = \ldots = \omega_{L_l;\theta} = \partial f_{l+1} = \ldots = \partial f_m = 0\},$$

where $L_j \in \mathbb{C}HS$ and f_k are smooth real functions satisfying

$$\omega_{L_1;\theta} \wedge \ldots \wedge \omega_{L_l;\theta} \wedge \partial f_{l+1} \wedge \ldots \wedge \partial f_m \neq 0, \quad \omega_{L_j;\theta}(L_x) = \theta([L, L_j])(x), \quad x \in S.$$

In Lecture 2 we proved that O is *complex-tangential* to S at p, i.e. $D_O \subset j_p^\infty \mathbb{C}HS \mod I(O)$, and the Levi form of S at p vanishes on T_pO. In particular, $T_pO \subset H_pS$ and all 1-forms $\omega_{L_j;\theta}$ vanish on T_pO.
On the other hand, the subbundle $E \oplus \overline{E}$ is tangent to the CR submanifold

$$M = \{x : f_s(x) = f_s(p),\ s = l+1, \ldots, m\} \subset S,$$

hence also any iterated commutator and the formal orbit O is tangent to $j_p^\infty M$, implying $T_pO \subset T_pM$, and therefore

$$H_p^{10} O \subset \{\partial f_{l+1} = \ldots = \partial f_m = 0\}.$$

Sketch of the proof — continuation

The obtained inclusion $H^{10}_p O \subset \{\partial f_{l+1} = \ldots = \partial f_m = 0\}$ together with vanishing $\omega_{L_j;\theta}$ above, implies

$$H^{10}_p O \subset \{\omega_{L_1;\theta} = \ldots = \omega_{L_l;\theta} = \partial f_{l+1} = \ldots = \partial f_m = 0\},$$

hence $H^{10}_p O \subset E_p$.

On the other hand, since formal vector fields in $j^\infty_p E$ are tangent to O and $E \subset H^{10} S$, ∞-jets of $(1,0)$ vector fields L with $L|_S$ in E are contained in the module D^{10}_O of formal $(1,0)$ vector fields tangent to O.

Since E is a subbundle, any vector in E_p can be extended to such vector field, it follows that $E_p \subset D^{10}_O(p)$. Combining with the above inclusion $H^{10}_p O \subset E_p$, we obtain

$$H^{10}_p O \subset D^{10}_O(p),$$

which is precisely our definition of O being CR. □

Complexifications of formal CR submanifolds are regular

As mentioned above, the intrinsic complexification of a real-analytic CR submanifold of $\mathbb{C}^n$ is *regular*, i.e. a complex submanifold of $\mathbb{C}^n$. For *formal CR submanifolds*, the analogue is a consequence of a lemma by Della Sala-Lamel-Juhlin:

Lemma

Let $O \subset \mathbb{C}^n$ be a formal CR submanifold with complexified ideal $\mathbb{C}I(O)$. Then formal holomorphic ideal

$$I' := \mathbb{C}I(O) \cap \mathbb{C}[[z]]$$

is a complex formal manifold ideal*, defining a complex formal submanifold.*

Corollary

Let S be pseudoconvex and $E \subset H^{10}S$ be a special subbundle of infinite Levi type at p. Then the complex orbit $\mathcal{O}_E^{\mathbb{C}}(p)$ *is a* complex formal submanifold*.*

A formal variant of a result of Diederich-Fornaess

To obtain a contradiction with finite regular type, we will show that the complex formal orbit $\mathcal{O}_E^{\mathbb{C}}(p)$ is *tangent of infinite order* to S.
An analogous statement in the real-analytic case is a key step by Diederich-Fornaess in their proof of the equivalence of finite type and zero holomorphic dimension, which together with Kohn's celebrated 1979 Acta paper yields subelliptic estimates under these conditions.

Theorem (*)

Let S be pseudoconvex, *$O \subset V \subset \mathbb{C}^n$ respectively real and complex formal submanifolds at p (let $p = 0$ for simplicity). Assume:*

1. *O is generic in V in the sense that $T_0 O + J T_0 O = T_0 V$;*
2. *O is* tangent of infinite order *to S, i.e. $O \subset j_0^\infty S$;*
3. *O is* complex-tangential *to $j_0^\infty S$, i.e. $D_O \subset j_0^\infty HS \mod I(O)$;*
4. *O is of* finite commutator type, *i.e.* $\mathrm{Lie}(D_O^{10} \oplus \overline{D_O^{10}})(0) = \mathbb{C} T_0 O$.

Then V is tangent to S of infinite order, i.e. $V \subset j_0^\infty S$.

Formal parametrizations of formal submanifolds

One obtains *formal immersions* from formal submanifolds by means of parametrization. By the *implicit function theorem for formal power series* (real or complex), after possible reordering coordinates, a manifold ideal I with *k real generators* also admits generators of the form

$$x_{j+m-k} - \phi_j(x'), \quad x' = (x_1, \ldots, x_{m-k}), \quad j = 1, \ldots, k,$$

and hence the formal power series map $A(x') := (x', \phi_1(x'), \ldots, \phi_k(x'))$ satisfies

$$\operatorname{rank} dA(0) = k, \quad F \circ A = 0 \text{ for all } F \in I.$$

Definition

Given a formal submanifold X with $I(X)$ having k real generators, a real formal power series map H with

$$\operatorname{rank} dH(0) = k, \quad F \circ A = 0 \text{ for all } F \in I(X).$$

is called a (formal) parametrization of X.

Approximation by maps into the given hypersurface

For the proof of *infinite order tangency*, we compare the Levi form of S with the complex hessian of defining function r along V. Our approach is to "*approximate*" a formal parametrization of V, or more generally, a *formal map A* into V with a *smooth map a* into S. We obtain a fine control of this approximation *modulo the ideal* of the pull-back $(j_0^\infty r) \circ A$:

Lemma (approximation lemma for maps)

Let $Y \subset \mathbb{R}^m$ be a formal hypersurface *defined by a principal ideal $I(Y)$. Let $R \in \mathbb{R}[[x]]$, $x = (x_1, \ldots, x_m)$, be a* generator *of $I(Y)$ with $dR(0) \neq 0$, and $A\colon (\mathbb{R}^s, 0) \to (\mathbb{R}^m, 0)$ any formal power series map. Then there is an* "approximating" formal power series *map $\widetilde{A}\colon (\mathbb{R}^s, 0) \to (\mathbb{R}^m, 0)$ satisfying*

1. $R \circ \widetilde{A} = 0$;
2. $\widetilde{A} - A = 0 \mod (R \circ A)$.

In case $Y = j_0^\infty S$ for a smooth real hypersurface $S \subset \mathbb{R}^m$ through 0, the approximating map can be chosen of the form $\widetilde{A} = j_0^\infty a$, where $a\colon (\mathbb{R}^s, 0) \to (S, 0)$ is a (germ of a) smooth map.

Approximation by tangent vector fields

Similar to approximation of maps, we *approximate formal* $(1,0)$ *vector fields* with smooth $(1,0)$ vector field tangent to S. We control the approximation by the ideal obtained from applying L to the Taylor series $j_0^\infty r$ of the defining function r.

Lemma (approximation lemma for vector fields)

Let $R \in \mathbb{C}[[z,\bar{z}]]$, $z = (z_1, \dots, z_n)$, be a real formal power series with $dR(0) \neq 0$ and $L = \sum a_j(z,\bar{z})\partial_{z_j}$ a $(1,0)$ formal vector field in $\mathbb{C}^n$. Then there exists an (approximating) $(1,0)$ formal vector field $\widetilde{L}$ in $\mathbb{C}^n$ satisfying:

1. $\widetilde{L}R = 0$;
2. $\widetilde{L} = L \mod (LR)$, *where (LR) is the principal ideal generated by LR.*

In case $R = j_0^\infty r$ for a smooth defining function r of a real hypersurface in $\mathbb{C}^n$, the approximating vector field $\widetilde{L}$ can be chosen of the form $\widetilde{L} = j_0^\infty \widetilde{l}$, where $\widetilde{l}$ is a germ at 0 of a smooth $(1,0)$ vector field in $\mathbb{C}^n$ with $\widetilde{l}r = 0$.

For a proof, set $\widetilde{L} := L - \frac{LR}{R_{z_n}}\partial_{z_n}$, $\widetilde{l} := l - \frac{lr}{r_{z_n}}\partial_{z_n}$.

Relative jets and contact orders

Since the real orbit O is already tangent to S of infinite order, we only estimate contact orders along $V \supset O$ transversally to O, called *relative contact orders*. More abstractly: let $X \subset Y$ be real formal submanifolds.

Definition

Let $X \subset Y \subset \mathbb{R}^m$ be *real formal submanifolds* given respectively by their manifold ideals $I(Y) \subset I(X) \subset \mathbb{R}[[x]]$.

1. Formal power series $F, G \in \mathbb{C}[[x]]$ are *(k, X, Y)-equivalent* if
$$F - G \in \mathbb{C}I(X)^{k+1} + \mathbb{C}I(Y)).$$
2. A *formal (k, X, Y)-jet* is a (k, X, Y)-equivalence class of in $\mathbb{C}[[x]]$.
3. The *relative contact order* at 0 between a hypersurface S with defining function r and the pair (X, Y) is
$$k = \min\{s : j_0^\infty r \in I(X)^s + I(Y)\} \in \mathbb{N}_{\geq 0} \cup \{\infty\}.$$

Positivity lemma on vanishing odd relative jets

The use of pseudoconvexity — by means of the following lemma:

Lemma (*positivity lemma*)

Let

$$f(x,y) \geq 0, \quad (x,y) \in \mathbb{R}^m_x \times \mathbb{R}^n_y, \quad m,n \geq 1,$$

be a real smooth nonnegative function in a neighborhood of 0*, whose Taylor series* $j_0^\infty f$ *at* 0 *satisfies*

$$j_0^\infty f \in I(\mathbb{R}^m_x)^k, \quad k \geq 0.$$

Assume that k is odd*. Then* $j_0^\infty f \in I(\mathbb{R}^m_x)^{k+1}$*.*

The proof is obtained by restrictions to real curves given by

$$(x,y) = (At, Bt^{2(b+1)}), \quad (A,B) \in \mathbb{R}^m_x \times \mathbb{R}^n_y, \quad t \in \mathbb{R},$$

with suitable sufficiently large b.

Pseudoconvexity implies even relative contact order

Corollary

Pseudoconvexity along with assumption (1)–(3) in Theorem $(*)$ *imply that the relative contact order* k *between* S *and* (O, V) *must be even.*

Since O is tangent of infinite order to S and complex-tangential,

$$D_V \subset j_0^\infty HS \mod I(O), \quad D_V := \{L : LI(V) \subset I(V)\},$$

hence for $R = j_p^\infty r$ (where r is defining function), $LR = 0 \mod I(O)$, hence $R \in I(O)^2 + I(V)$, i.e. $k \geq 2$. *Assume k is odd*. Using approximations by maps into hypersurface and by tangent vector fields, along with pseudoconvexity and positivity lemma yields

$$\partial\bar{\partial}R(L, \overline{L}) \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V), \quad L \in D_V^{10}, \quad \mathcal{D}_V^{10} = \mathbb{C}D_V \cap \Gamma(H^{10}\mathbb{C}^n).$$

Since O is generic in V, by *Cartan's lemma*, it can be shown that

$$L^2L^1R \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V), \quad L^2, L^1 \in \mathbb{C}D_V,$$

and hence that $R \in I(O)^{k+1} + I(V)$, a contradiction with choice of k. □

Details: relative order for the Levi form

Let A be a joint parametrization for O, V, hence $R \circ A \in I(\mathbb{R}^{s'})^k$. By the *approximation lemmas*, there exists an approximation by a smooth map germ $a\colon (\mathbb{R}^{s'} \times \mathbb{R}^{s''}, 0) \to (S, 0)$ with

$$j_0^\infty a = A \mod I(\mathbb{R}^{s''})^k.$$

and a smooth $(1,0)$ vector field germ l at 0 in $\mathbb{C}^n$ with

$$l|_S \in H^{10}S, \quad j_0^\infty l = L \mod \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V).$$

we obtain

$$j_0^\infty(\partial\bar\partial r(l, \bar l) \circ a) - \partial\bar\partial R(L, \bar L) \circ A \in I(\mathbb{R}^{s'})^{k-1}.$$

Since S is *pseudoconvex*, apply *positivity lemma* to $\partial\bar\partial r(l, \bar l)|_S \geq 0$:

$$j_0^\infty(\partial\bar\partial r(l, \bar l) \circ a) \in I(\mathbb{R}^{s'})^{k-1} \implies \partial\bar\partial R(L, \bar L) \circ A \in I(\mathbb{R}^{s'})^{k-1}.$$

when k is odd. Since A is the joint parametrization of (O, V),

$$\partial\bar\partial R(L, \bar L) \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V).$$

Details: Use of Cartan's formula

We have shown: $\partial\bar{\partial}R(L,\overline{L}) \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V)$. By polarization,

$$\partial\bar{\partial}R(L^2,\overline{L^1}) \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V), \quad L^2, L^1 \in D_V^{10}.$$

By *Cartan's formula* applied to the left-hand side,

$$L^2\overline{\partial}R(\overline{L^1}) - \overline{L^1}\,\overline{\partial}R(L^2) - \overline{\partial}R([L^2,\overline{L^1}]) \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V).$$

The first term equals $L^2\overline{L^1}R$, the second term vanishes since $\overline{\partial}R$ is a $(0,1)$ form, and the last term belongs to $\mathbb{C}I(O)^{k-1} + \mathbb{C}I(V)$, since $R \in I(O)^k + I(V)$ and $[L^2,\overline{L^1}], J[L^2,\overline{L^1}] \in \mathbb{C}D_V$, hence

$$L^2\overline{L^1}R \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V), \quad L^2, L^1 \in D_V^{10}.$$

Since differentiation along O does not reduce the order, we can prove that, whenever $L^j \in \mathbb{C}D_O$ for either $j=1$ or $j=2$,

$$L^2L^1R \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V), \quad L^2, L^1 \in \mathbb{C}D_V.$$

Details: the relative contact order k must be even

We have shown:

$$L^2\overline{L^1}R \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V), \quad L^2, L^1 \in D_V^{10},$$

and whenever $L^j \in \mathbb{C}D_O$ for either $j = 1$ or $j = 2$,

$$L^2 L^1 R \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V), \quad L^2, L^1 \in \mathbb{C}D_V.$$

By assumption (1) that O is *generic* in V (i.e. $T_0 O + JT_0 O = T_0 V$),

$$\mathbb{C}D_O \cap \mathbb{C}D_V + D_V^{10} = \mathbb{C}D_O \cap \mathbb{C}D_V + \overline{D_V^{10}} = \mathbb{C}D_V \mod \mathbb{C}I(V),$$

which in combination with the above yields

$$L^2 L^1 R \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V), \quad L^2, L^1 \in \mathbb{C}D_V,$$

whenever k is odd. Since O is *complex-tangential*, $k \geq 2$, and the above can be shown (by integration) to imply

$$R \in I(O)^{k+1} + I(V),$$

i.e. the relative contact order is $\geq k+1$, a contradiction with our choice of $k = \min\{s : R \in I(O)^s + I(V)\}$. Hence k must be *even* as claimed.

Supertangent vector fields

Our next new tool in the proof of Theorem $(*)$ is the notion that we call *supertangent* and *complex-supertangent* vector fields to a (formal) hypersurface, which applied to a defining function of the hypersurface *do not reduce the relative contact order*. As before, we write $R = j_o^\infty r$, where r is a smooth local defining function of S.

Definition

Let k be the relative contact order at $p = 0$ between S and the pair (O, V). A (complex) formal vector field $L \in \mathbb{C}D_V$ is said to be

1. *supertangent* to S (or $j_0^\infty S$) if $LR \in \mathbb{C}I(O)^k + \mathbb{C}I(V)$;
2. *complex-supertangent* to S (or $j_0^\infty S$) if both L and JL are supertangent, where J is the complex structure.

Since R is real, a complex vector field L is supertangent whenever both real and imaginary parts of it are supertangent.

$(1,0)$ vector fields along O are complex-supertangent

An immediate source of supertangent vector fields is provided by the following simple lemma:

Lemma

Any vector field $L \in (D_O^{10} \oplus \overline{D_O^{10}}) \cap \mathbb{C}D_V$ *is* complex-supertangent.

Proof.

Given L as in the lemma, it follows that

$$L, JL \in \mathbb{C}D_O \cap \mathbb{C}D_V,$$

i.e., L and JL leave both ideals $\mathbb{C}I(O)$, $\mathbb{C}I(V)$ invariant. Since $R \in I(O)^k + I(V)$, it follows that

$$LR, (JL)R \in I(O)^k + I(V),$$

proving the desired conclusion. □

Lie algebra property of complex-supertangent vector fields

Proposition

Assume the contact order k between S and (O, V) is *even*. Let $L^2, L^1 \in \mathbb{C}D_O \cap \mathbb{C}D_V$ be complex-supertangent vector fields, i.e.

$$L^j R, (JL^j)R \in \mathbb{C}I(O)^k + \mathbb{C}I(V), \quad j = 1, 2.$$

Then their *commutator* $[L^2, L^1]$ *is also complex-supertangent*, i.e.

$$[L^2, L^1]R, (J[L^2, L^1])R \in \mathbb{C}I(O)^k + \mathbb{C}I(V).$$

I.e. complex-supertangent vector fields in $\mathbb{C}D_O \cap \mathbb{C}D_V$ form a *Lie algebra*.

A key step is higher than expected relative order for complex hessian:

Lemma (higher order of hessian along supertangent v.f. for even k)

$$\partial\bar{\partial}R(L, \bar{L}) \in I(O)^k + I(V)$$

holds for any supertangent *vector field* $L \in D_V^{10}$.

Proof of Lemma on higher vanishing order for the hessian

The proof consists of approximating $\partial\bar{\partial}R(L,\bar{L})$ with a complex hessian $\partial\bar{\partial}r(l,\bar{l})$ along a $(1,0)$ vector field l on S, then using *pseudoconvexity* along with *positivity Lemma* above. By Cartan's formula,

$$\partial\bar{\partial}R(L,\bar{L}) = L\bar{\partial}R(\bar{L}) - \bar{L}\,\bar{\partial}R(L) - \bar{\partial}R([L,\bar{L}]). \qquad (*)$$

Since L is supertangent and R is real,

$$\bar{\partial}R(\bar{L}) = \overline{LR} \in \mathbb{C}I(O)^k + \mathbb{C}I(V),$$

hence the first term in $(*)$ belongs to $\mathbb{C}I(O)^{k-1} + \mathbb{C}I(V)$. The second term vanishes since $L \in D_V^{10}$, while the last term again belongs to $\mathbb{C}I(O)^{k-1} + \mathbb{C}I(V)$, since $[L,\bar{L}], J[L,\bar{L}] \in D_V$. Hence

$$\partial\bar{\partial}R(L,\bar{L}) \in \mathbb{C}I(O)^{k-1} + \mathbb{C}I(V). \qquad (1)$$

Since S is *pseudoconvex*, we can approximate the left-hand side with a nonnegative function modulo $\mathbb{C}I(V)$, and obtain higher relative order from the positivity Lemma.

Proof of the Lie algebra property

By Cartan's formula applied to the 1-forms $\overline{\partial}R$, ∂R,

$$\begin{aligned}\partial\overline{\partial}R(L^2, L^1) &= L^2\overline{\partial}R(L^1) - L^1\overline{\partial}R(L^2) - \overline{\partial}R([L^2, L^1]),\\ \overline{\partial}\partial R(L^2, L^1) &= L^2\partial R(L^1) - L^1\partial R(L^2) - \partial R([L^2, L^1]).\end{aligned} \qquad (*)$$

Write $L^j = L^j_{10} + L^j_{01}$, $j = 1, 2$, as sum of $(1, 0)$ and $(0, 1)$ components, then both components L^j_{10}, L^j_{01} are *supertangent* and

$$\partial R(L^j) = \partial R(L^j_{10}),\ \overline{\partial}R(L^j) = \partial R(L^j_{01}) \in \mathbb{C}I(O)^k + \mathbb{C}I(V). \qquad (**)$$

By Lemma on higher vanishing order along with polarization,

$$\partial\overline{\partial}R(L^2, L^1) = \partial\overline{\partial}R(L^2_{10}, L^1_{01}) - \partial\overline{\partial}R(L^2_{01}, L^1_{10}) \in \mathbb{C}I(O)^k + \mathbb{C}I(V),$$

i.e. left-hand sides of $(*)$ are in $\mathbb{C}I(O)^k + \mathbb{C}I(V)$. Applying L^j to $(**)$ and using our assumption $L^j \in \mathbb{C}D_O \cap \mathbb{C}D_V$, we compute, using $(*)$:

$$L^2\overline{\partial}R(L^1), L^1\overline{\partial}R(L^2), L^2\partial R(L^1), L^1\partial R(L^2) \in \mathbb{C}I(O)^k + \mathbb{C}I(V),$$

$$\overline{\partial}R([L^2, L^1]), \partial R([L^2, L^1]) \in \mathbb{C}I(O)^k + \mathbb{C}I(V),$$

which implies the desired higher order vanishing. □

Achieving contradiction with finiteness of k

We now use the last assumtion $\mathrm{Lie}(D_O^{10} \oplus \overline{D_O^{10}})(0) = \mathbb{C}T_0 O$ to show that *all vector fields* in $\mathbb{C}D_O \cap \mathbb{C}D_V$ are in fact *complex-supertangent*:

Corollary

Assume k is even. Then all vector fields in $\mathbb{C}D_V$ are supertangent.

Proof. By the Lie algebra property, *iterated commutators* of vector fields in $(D_O^{10} \oplus \overline{D_O^{10}}) \cap \mathbb{C}D_V$ are complex-supertangent. By the genericity assumption $T_0 O + JT_0 O = T_0 V$, it can be shown that $\mathbb{C}D_V$ is *spanned by supertangent vector fields* over the ring $\mathbb{C}[[z, \overline{z}]]$ modulo $\mathbb{C}I(V)$. □

Proof of Theorem on infinite contact order between S and V.

Assume by contradiction that k is finite. We saw that k must be even. By the above Corollary, $LR \in I(O)^k + I(V)$ for all $L \in \mathbb{C}D_V$. It can be shown that the latter implies higher order vanishing $R \in \mathbb{C}I(O)^{k+1} + \mathbb{C}I(V)$, contradicting the choice of k. □

I.e. complex orbit V has *infinite contact order* violating *finite regular type*.