New tools and conditions for global regularity of the $\bar{\partial}$-Neumann operator

Part 2 - Commutator type and formal orbits

Dmitri Zaitsev zaitsev@maths.tcd.ie

Trinity College Dublin
1. *smooth* means always $C^\infty$;

2. $S \subset \mathbb{C}^{n+1}$, $n \geq 1$, (or $S \subset \mathbb{C}^n$, $n \geq 2$) is a smooth real hypersurface;

3. a *local defining function* $r$ of $S$ in a neighborhood $U$ of $p \in S$ is any smooth real function with

$$S \cap U = \{ r = 0 \}$$

and $dr \neq 0$ at every point of $U$;

4. any two local defining functions $r_1, r_2$ in $U$ differ by a *nonzero smooth function factor*:

$$r_2(x) = a(x)r_1(x), \quad a(x) \neq 0.$$
Notation: tangent bundles

1. $TS$ is the **real tangent bundle**;
2. $\mathbb{C}TS = \mathbb{C} \otimes_{\mathbb{R}} TS$ is the **complexified tangent bundle**;
3. $H^{10}S = \{X \in \mathbb{C}TS : \partial r(X) = 0\}$ is the $(1, 0)$ bundle;
4. $H^{01}S = \{X \in \mathbb{C}TS : \bar{\partial} r(X) = 0\}$ is the $(0, 1)$ bundle;
5. $HS = \text{Re} H^{10}S = \text{Re} H^{01}S \subset TS$ is the **complex tangent bundle**;
6. We have the standard relations:

\[ H^{01}S = \overline{H^{10}S}, \quad \mathbb{C}HS = H^{10}S \oplus \overline{H^{10}S}. \]
Shown in Lecture 1:

finite Levi type $\quad \rightarrow \quad$ finite tower multitype

$\quad \rightarrow \quad$ generalized stratifications

Our next goal will be to extend the first implication to another finiteness condition based on the **commutator type**.
Hypersurfaces with subbundles of finite commutator type

The following type notion goes back to Kohn’s 1972 JDG paper:

**Definition**

The **commutator type** $t(E, p) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ at $p \in S$ of a subbundle $E \subset H^{10}S$ is

$$\min\{t \geq 2 : \exists L^t, \ldots, L^1 \in \Gamma(E) \cup \Gamma(E), [L^m, \ldots, [L^2, L^1] \ldots](p) \not\in \mathbb{C}HS\}.$$  

Our next goal is to show that finite commutator type for all subbundles implies finite tower multitype:

**Theorem**

*For a pseudoconvex smooth hypersurface $S \subset \mathbb{C}^{n+1}$, assume $t(E, p) < \infty$ for any smooth subbundle $E \subset H^{10}S$ of rank 1 and any $p \in S$. Then $S$ is of finite tower multitype at $p$.***

This can be combined with previously shown implications to conclude compactness and global regularity in the $\bar{\partial}$-Neumann problem.
Recall: special subbundles

We prove a stronger more refined version of the above theorem, where the commutator type finiteness \( t(E, p) < \infty \) is only required to check for certain \textit{special subbundles} \( E \):

**Definition (special subbundles)**

A complex subbundle \( E \subset H^{10} S \) is called \textit{special} if it can be defined by

\[
E = \{ \xi \in H^{10} S : \omega_1(\xi) = \ldots = \omega_l(\xi) = 0 \}, \quad \omega_1 \wedge \cdots \wedge \omega_l \neq 0 \text{ on } (H^{10} S)^l,
\]

where each \( \omega_j, j = 1, \ldots, l \), is the \( \theta \)-dual \textit{1-form} \( \omega_j = \omega_{L_j^{t_j}, \ldots, L_j^1} \) for some \( t_j \geq 1 \) and vector fields \( L_j^{t_j}, \ldots, L_j^1 \in \Gamma(H^{10} S) \cup \Gamma(H^{10} S) \).

**Theorem (finite commutator type only for special subbundles)**

Assume \( t(E, p) < \infty \) for any \textit{special subbundle} \( E \subset H^{10} S \) of rank \( \geq 1 \). Then \( S \) is of \textit{finite tower multitype} at \( p \).

Note that \( t(E, p) < \infty \) implies \( t(E', p) < \infty \) for any subbundle \( E' \subset E \).
Various notions of orbits for sets of vector fields

To study commutators of vector fields geometrically, we employ constructions of their integral manifolds called *orbits*.

1. **Sussman orbits**: the tangent space at a fixed point cannot be recovered from values of iterated commutators.

2. **Nagano orbits**: only defined in the real-analytic category.

Instead, we use *formal orbits* defined in terms of formal Taylor series of arbitrary smooth vector fields.

**Definition**

Let $\mathcal{L}$ be a set of smooth vector fields in a neighborhood of $p$ in $\mathbb{C}^n$, and $\mathcal{R} \subset \mathbb{C}[[z - p, \overline{z} - p]]$ a subring of formal power series. The *formal orbit* $O = \mathcal{O}_\mathcal{L}^\mathcal{R}(p)$ is the formal variety given by the *formal power series ideal*

$$I(O) = \{ f \in \mathcal{R} : L^t \cdots L^1 f(p) = 0, \: L^t, \ldots, L^1 \in \mathcal{L}, \: t \geq 0 \}.$$  

Each $L^t \cdots L^1 f$, calculated via formal power series expansions, has the form

$$\sum_{\alpha, \beta} c_{\alpha \beta} (z - p)^\alpha (\overline{z} - p)^\beta \in \mathbb{C}[[z - p, \overline{z} - p]].$$
Real and complex orbits of smooth subbundles

For a subbundle $E \subset H^{10} S$, consider the set $\mathcal{L} = \mathcal{L}(E)$ of smooth vector fields in a neighborhood of $p$ in $\mathbb{C}^n$, whose restrictions to $S$ are either in $E$ or its conjugate:

$$\mathcal{L} = \mathcal{L}(E) = \{ L : L|_S \in \Gamma(E) \cup \Gamma(\overline{E}) \}.$$ 

Note that the condition $L^t \cdots L^1 f(p) = 0$ for $L^j \in \mathcal{L}(E)$ only depends on the restrictions $L^j|_S$. We consider two subrings

$$\mathcal{R}_\mathbb{R} := \{ f \in \mathbb{C}[[z - p, \overline{z} - p]] : \overline{f} = f \} \quad \text{and} \quad \mathcal{R}_\mathbb{C} := \mathbb{C}[[z - p]].$$

**Definition**

The **real formal orbit** (resp. **complex formal orbit**) of $E$ is given by

$$\mathcal{O}_{\mathcal{L}(E)}^{\mathcal{R}_\mathbb{R}} := \mathcal{O}_E^{\mathcal{R}_\mathbb{R}} \quad \text{(resp.} \quad \mathcal{O}_E^{\mathcal{R}_\mathbb{C}} := \mathcal{O}_E^{\mathcal{R}_\mathbb{C}})).$$

We shall show in our cases of interest, both orbits are so-called **formal submanifolds**.
Formal submanifolds and manifold ideals

It is a consequence of a “formal variant of Nagano’s theorem” by Baouendi-Ebenfelt-Rothschild (an adaptation to the formal case of my unpublished proof of Nagano’s theorem), that the real formal orbit $\mathcal{O}_{\mathbb{R}}^E$ is always a so-called real formal submanifold.

The definition is an adaptation of the definition of a germ at $p$ of a smooth submanifold $M \subset \mathbb{C}^n$, whose ideal $I(M)$ in the ring of smooth function germs at $p$ is generated by a finite subset with linearly independent differentials at $p$:

**Definition**

1. A manifold ideal $I$ in a ring $\mathcal{R} \subset \mathbb{C}[[z - p, \overline{z} - \overline{p}]]$ is one generated by $k$ elements $F_1, \ldots, F_k \in I$ with linearly independent differentials at $p$.

2. A real (resp. complex) formal submanifold $O$ at $p$ in $\mathbb{C}^n$ is one defined by a manifold ideal $I = I(O)$ in $\mathcal{R}_{\mathbb{R}}$ (resp. $\mathcal{R}_{\mathbb{C}}$).

Note that, unlike germs of smooth functions, we can only evaluate differentials of formal power series at the base point $p$. 
A small abuse of notation

We assume the contact form $\theta$ on $S$ is extended to a neighborhood of $p$ in $\mathbb{C}^n$ (our construction will be independent of this extension).

Notation for formal Taylor series of smooth functions

Given a smooth function $f$ in a neighborhood of $p$ in $\mathbb{C}^n$, we write $j_p^\infty f \in \mathbb{C}[[z - p, \overline{z} - p]]$ for its formal Taylor series at $p$.

As a direct consequence of our definition of the real orbit, we obtain:

Corollary

If $E$ has infinite Levi type $c(E, p) = \infty$, i.e. $L^m \cdots L^3 \theta([L^2, L^1])(p) = 0$ for all $L^j \in \Gamma(E) \cup \Gamma(\overline{E})$, then

$$j_p^\infty \theta([L^2, L^1]) \in \mathbb{C}I(O_E^\mathbb{R})$$

for all complex vector fields $L^2, L^1 \in \mathcal{L}(E)$. 

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Weighted expansion of nonnegative functions

Our next goal is to use pseudoconvexity to obtain some stronger conclusions. We need the following well-known lemma:

**Lemma (positivity of the lowest weight component)**

Let \( f \geq 0 \) be a nonnegative smooth function in a neighborhood of 0 in \( \mathbb{R}^m \). Fix a collection of positive weights \( \mu_j > 0, j = 1, \ldots, m \), and number \( k \geq 0 \), and consider a decomposition

\[
f = f_k + f_{\succ k},
\]

where \( f_k \) is weighted homogeneous of degree \( k \), while the Taylor expansion of \( f_{\succ k} \) at 0 consists of terms of weight \( > k \). Then \( f_k \geq 0 \).

**Proof** The conclusion is obtained by taking the limit of

\[
t^{-k} f(t^{\mu_1} x_1, \ldots, t^{\mu_m} x_m) \geq 0
\]

as \( t \to 0 \) for each fixed \( (x_1, \ldots, x_m) \in \mathbb{R}^m \).
Formal parametrizations of formal submanifolds

By the *implicit function theorem for formal power series*, after possible reordering coordinates, a manifold ideal $I$ with *k real generators* also admits generators of the form

$$x_{j+m-k} - \phi_j(x'), \quad x' = (x_1, \ldots, x_{m-k}), \quad j = 1, \ldots, k,$$

and hence the real formal power series map

$$A(x') := (x', \phi_1(x'), \ldots, \phi_k(x'))$$

satisfies

$$\text{rank } dA(0) = k, \quad F \circ A = 0 \text{ for all } F \in I.$$

**Definition**

Given a formal submanifold $O$ with $I(O)$ having $k$ real generators, a real formal power series map $H$ with

$$\text{rank } dH(0) = k, \quad F \circ A = 0 \text{ for all } F \in I(O).$$

is called a *(formal) parametrization of $O$.*
Pseudoconvexity $\iff$ Levi null cone coincides with kernel

**Notation**

1. Write $j^\infty_p S$ for the formal submanifold defined by the ideal of all Taylor series of germs of smooth functions vanishing on $S$.
2. Write $O \subset O'$ for formal submanifolds $O, O'$ whenever $I(O) \supset I(O')$.

In particular, for a subbundle $E \subset \mathbb{C}TM$, one has $O^E_E \subset j_0^\infty S$.

A well-known result from Linear Algebra for a *positive semi-definite* hermitian form $h(\cdot, \cdot)$ asserts that $h(L, L) = 0$ for some $L$ implies $h(L, L') = 0$ for any $L'$. We show a formal variant of this statement:

**Proposition (Levi null cone vs Levi kernel)**

Let $S \subset \mathbb{C}^n$ be pseudoconvex, $0 \in S$, and $O \subset j_0^\infty S$ a real formal submanifold. Let $L, L'$ be smooth complex vector fields in a neighborhood of $0$ in $\mathbb{C}^n$ with $L|_S, L'|_S \in H^{10} S$. Then

$$j_0^\infty \theta([L, \bar{L}]) \in \mathbb{C}I(O) \implies j_0^\infty \theta([L, \bar{L}']) \in \mathbb{C}I(O).$$
Proof of the Levi cone vs kernel proposition

Pseudoconvexity with suitable choice of $\theta$ implies for all $c \in \mathbb{C}$:

$$\theta([L + cL', \overline{L} + c\overline{L'}]) = \theta([L, \overline{L}]) + 2\text{Re}(\overline{c}\theta([L, \overline{L'}])) + c\overline{c}\theta([L', \overline{L'}]) \geq 0$$

Since $O \subset j^\infty_p S$, there exists a smooth map germ $\gamma: (\mathbb{R}^q, 0) \to (S, 0)$ such that $j^\infty_0 \gamma$ is a formal parametrization of $O$. Take pullbacks under $\gamma$:

$$\gamma^*\theta([L, \overline{L}]) + 2\text{Re}(\overline{c}\gamma^*\theta([L, \overline{L'}])) + c\overline{c}\gamma^*\theta([L', \overline{L'}]) \geq 0'$$

Since $j^\infty_0 \theta([L, \overline{L'}]) \in CI(O) \iff \gamma^*\theta([L, \overline{L'}])$ vanishes of infinite order, assume by contradiction that order $s$ to be finite. Assign weights 1 to $(t_1, \ldots, t_q) \in \mathbb{R}^q$ and $\mu = s + 1$ to $(\text{Re}, \text{Im} c)$. The middle term has the lowest weight term $2\text{Re}(\overline{c}\gamma^*\theta([L, \overline{L'}]))_{s+\mu} \neq 0$, $c\overline{c}\gamma^*\theta([L', \overline{L'}])$ has weight $\geq 2\mu > s + \mu$, and $\gamma^*\theta([L, \overline{L}])$ vanishes of infinite order when $j^\infty_p \theta([L, \overline{L}]) \in CI(O)$. Therefore, the lowest weight term satisfies

$$2\text{Re}(\overline{c}\gamma^*\theta([L, \overline{L'}]))_{s+\mu} \geq 0$$

by positivity lemma. Since $c$ is arbitrary, the left-hand side must vanish, which contradicts the above conclusion in red, completing the proof.
Complex-tangential property for real orbits

Notation

1. For a subbundle $E$, denote by $j_p^\infty E$ the space of all formal Taylor expansions of complex vector fields in whose restrictions to $S$ are in $E$.
2. For a formal submanifold $O$, denote by $D_O$ the module of all formal vector fields $L$ tangent to $O$ in the sense of $L(I(O)) \subset I(O)$.

Corollary

Let $S$ be pseudoconvex and a subbundle $E \subset H^{10}S$ be of infinite Levi type at $p \in S$. Then $O = O^E_{\mathbb{R}}(p)$ satisfies:

1. $[j_p^\infty (E \oplus \bar{E}), j_p^\infty \mathbb{C}HS] \subset j_p^\infty \mathbb{C}HS \mod \mathbb{C}I(O)$.
2. $[D_O, j_p^\infty \mathbb{C}HS] \subset j_p^\infty \mathbb{C}HS \mod \mathbb{C}I(O)$.
3. The orbit $O$ is complex-tangential to $j_p^\infty S$ in the sense that $D_O \subset j_p^\infty \mathbb{C}HS \mod \mathbb{C}I(O)$. 
Proof of (1)

Consider formal vector fields in \( j^\infty_p E \) and \( j^\infty_p H^{10}S \) which are Taylor series at \( p \) of smooth vector fields \( L, L' \) whose restrictions to \( S \) are in \( E \) and \( H^{10}S \) respectively. Since \( E \) is of infinite Levi type,

\[
j^\infty_p \theta([L, L]) \in \mathcal{C}\mathcal{I}(O),
\]

and the “Levi null cone vs kernel proposition” implies

\[
j^\infty_p \theta([L, L']) \in \mathcal{C}\mathcal{I}(O).
\]

The above holds also for \( L' \) with \( L'|_S \in H^{01}S \), since \( H^{10}S \) is an involutive distribution in \( \mathbb{C}TS \) by the integrability of the CR structure. Since \( \theta \) is a contact form,

\[
[j^\infty_p E, j^\infty_p \mathcal{C}HS] \subset j^\infty_p \mathcal{C}HS \mod \mathcal{C}\mathcal{I}(O),
\]

from which (1) follows, since the right-hand side is invariant under conjugation.
Proof of (2) and (3)

To show (2), observe that (1) along with Jacobi identity implies that

$$[L, j_p^\infty \mathbb{C}HS] \subset j_p^\infty \mathbb{C}HS \mod \mathbb{C}I(O)$$

for any $L$ which is an iterated Lie bracket of formal vector fields in $j_p^\infty (E \oplus \overline{E})$. Since $O = O_E^\mathbb{R}(p)$ is the orbit of the Lie algebra $\mathfrak{g} = \mathfrak{g}_E^\mathbb{R}$ spanned by the real parts $\text{Re}L = \frac{1}{2}(L + \overline{L})$ of iterated Lie brackets, it follows that

$$[\mathfrak{g}, j_p^\infty \mathbb{C}HS] \subset j_p^\infty \mathbb{C}HS \mod \mathbb{C}I(O).$$

It can be shown that the orbit $O$ satisfies $D_O = \mathfrak{g} \mod I(O)$, proving (2).

Now, since $E \oplus \overline{E} \subset \mathbb{C}HS$, by repeatedly using (1), we conclude that any iterated Lie bracket of vector fields in $j_p^\infty (E \oplus \overline{E})$ is contained in $j_p^\infty \mathbb{C}HS$ modulo $\mathbb{C}I(O)$. As before, using the relation $D_O = \mathfrak{g} \mod I(O)$ completes the proof of (3).
Finite commutator type implies finite tower multitype

We use the above Corollary, part (3) to conclude the inclusion of tangent spaces

\[ T_p O \subset H_p S, \]

where the \( T_p O \) is defined by the vanishing of all \( df(p) \) with \( f \in I(O) \). We now restate and prove the main theorem of this lecture:

**Theorem (finite commutator type only for special subbundles)**

Assume \( t(E, p) < \infty \) for any special subbundle \( E \subset H^{10} S \) of rank 1. Then \( S \) is of finite tower multitype at \( p \).

**Proof.** It suffices to show finite Levi type at \( p \) for any special subbundle \( E \subset H^{10} S \) of rank \( \geq 1 \), since that implies finite tower multitype at \( p \) as we proved in Lecture 1. Assume by contradiction, there is a special subbundle \( E \) of rank \( \geq 1 \), which is of infinite Levi type at \( p \). Then the above inclusion of tangent spaces holds, and since formal Taylor series of any iterated commutator \([L^m, \ldots, [L^2, L^1]\ldots]\) for \( L^m, \ldots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E}) \), are tangent to the formal orbit \( O = \mathcal{O}_E^R(p) \), their values at \( p \) stay in \( \mathbb{C}H_p S \), contradicting \( t(E, p) < \infty \).