New tools and conditions for global regularity of the $\bar{\partial}$-Neumann operator

Part 2 - Commutator type and formal orbits

Dmitri Zaitsev *zaitsev@maths.tcd.ie*

Trinity College Dublin

Notation: local defining functions

1. *smooth* means always C^∞;
2. $S \subset \mathbb{C}^{n+1}$, $n \geq 1$, (or $S \subset \mathbb{C}^n$, $n \geq 2$) is a smooth real hypersurface;
3. a *local defining function* r of S in a neighborhood U of $p \in S$ is any smooth real function with

$$S \cap U = \{r = 0\}$$

   and $dr \neq 0$ at every point of U;
4. any two local defining functions r_1, r_2 in U differ by a *nonzero smooth function factor*:

$$r_2(x) = a(x) r_1(x), \quad a(x) \neq 0.$$

Notation: tangent bundles

1. TS is the *real tangent bundle*;
2. $\mathbb{C}TS = \mathbb{C} \otimes_{\mathbb{R}} TS$ is the *complexified tangent bundle*;
3. $H^{10}S = \{X \in \mathbb{C}TS : \partial r(X) = 0\}$ is the $(1,0)$ bundle;
4. $H^{01}S = \{X \in \mathbb{C}TS : \bar{\partial} r(X) = 0\}$ is the $(0,1)$ bundle;
5. $HS = \mathrm{Re} H^{10}S = \mathrm{Re} H^{01}S \subset TS$ is the *complex tangent bundle*;
6. We have the standard relations:

$$H^{01}S = \overline{H^{10}S}, \quad \mathbb{C}HS = H^{10}S \oplus \overline{H^{10}S}.$$

Motivation: from finiteness conditions to global regularity

finiteness conditions $\Longrightarrow$ generalized stratifications

$\Downarrow$

condition R $\qquad$ property (P_q)

$\Uparrow$ $\qquad$ $\Downarrow$

global regularity $\Longleftarrow$ compactness

Shown in Lecture 1:

finite Levi type $\Longrightarrow$ finite tower multitype

$\Downarrow$

generalized stratifications

Our next goal will be to extend the first implication to another finiteness condition based on the *commutator type*.

Hypersurfaces with subbundles of finite commutator type

The following type notion goes back to Kohn's 1972 JDG paper:

Definition

The commutator type $t(E,p) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ at $p \in S$ of a subbundle $E \subset H^{10}S$ is

$$\min\{t \geq 2 : \exists L^t, \ldots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E}), [L^m, \ldots, [L^2, L^1] \ldots](p) \notin \mathbb{C}HS\}.$$

Our next goal is to show that finite commutator type for all subbundles implies finite tower multitype:

Theorem

For a pseudoconvex smooth hypersurface $S \subset \mathbb{C}^{n+1}$, assume $t(E,p) < \infty$ for any smooth subbundle $E \subset H^{10}S$ of rank 1 and any $p \in S$. Then S is of finite tower multitype at p.

This can be combined with previously shown implications to conclude compactness and global regularity in the $\bar{\partial}$-Neumann problem.

Recall: special subbundles

We prove a stronger more refined version of the above theorem, where the commutator type finiteness $t(E, p) < \infty$ is only required to check for certain *special subbundles* E:

Definition (special subbundles)

A complex subbundle $E \subset H^{10}S$ is called special if it can be defined by

$E = \{\xi \in H^{10}S : \omega_1(\xi) = \ldots = \omega_l(\xi) = 0\}, \quad \omega_1 \wedge \cdots \wedge \omega_l \neq 0$ on $(H^{10}S)^l$,

where each ω_j, $j = 1, \ldots, l$, is the *θ-dual 1-form* $\omega_j = \omega_{L_j^{t_j}, \ldots, L_j^1}$ for some $t_j \geq 1$ and vector fields $L_j^{t_j}, \ldots, L_j^1 \in \Gamma(H^{10}S) \cup \Gamma(\overline{H^{10}S})$.

Theorem (finite commutator type only for special subbundles)

Assume $t(E, p) < \infty$ for any special subbundle *$E \subset H^{10}S$ of rank ≥ 1. Then S is of* finite tower multitype *at p.*

Note that $t(E, p) < \infty$ implies $t(E', p) < \infty$ for any subbundle $E' \subset E$.

Various notions of orbits for sets of vector fields

To study commutators of vector fields geometrically, we employ constructions of their integral manifolds called *orbits*.

1. *Sussman orbits*: the tangent space at a fixed point cannot be recovered from values of iterated commutators.
2. *Nagano orbits*: only defined in the real-analytic category.

Instead, we use *formal orbits* defined in terms of formal Taylor series of arbitrary smooth vector fields.

Definition

Let $\mathcal{L}$ be a set of smooth vector fields in a neighborhood of p in $\mathbb{C}^n$, and $\mathcal{R} \subset \mathbb{C}[[z-p, \overline{z-p}]]$ a subring of formal power series. The formal orbit $O = \mathcal{O}_{\mathcal{L}}^{\mathcal{R}}(p)$ is the formal variety given by the *formal power series ideal*

$$I(O) = \{f \in \mathcal{R} : L^t \cdots L^1 f(p) = 0,\ L^t, \ldots, L^1 \in \mathcal{L},\ t \geq 0\}.$$

Each $L^t \cdots L^1 f$, calculated via formal power series expansions, has the form

$$\sum_{\alpha\beta} c_{\alpha\beta}(z-p)^\alpha(\overline{z-p})^\beta \in \mathbb{C}[[z-p, \overline{z-p}]].$$

Real and complex orbits of smooth subbundles

For a subbundle $E \subset H^{10}S$, consider the set $\mathcal{L} = \mathcal{L}(E)$ of smooth vector fields in a neighborhood of p in $\mathbb{C}^n$, whose restrictions to S are either in E or its conjugate:

$$\mathcal{L} = \mathcal{L}(E) = \{L : L|_S \in \Gamma(E) \cup \Gamma(\overline{E})\}.$$

Note that the condition $L^t \cdots L^1 f(p) = 0$ for $L^j \in \mathcal{L}(E)$ only depends on the restrictions $L^j|_S$. We consider two subrings

$$\mathcal{R}_{\mathbb{R}} := \{f \in \mathbb{C}[[z-p, \overline{z-p}] : \bar{f} = f\} \quad \text{and} \quad \mathcal{R}_{\mathbb{C}} := \mathbb{C}[[z-p]].$$

Definition

The *real formal orbit* (resp. *complex formal orbit*) of E is given by

$$\mathcal{O}_E^{\mathbb{R}} := \mathcal{O}_{\mathcal{L}(E)}^{\mathcal{R}_{\mathbb{R}}} \quad (\text{resp. } \mathcal{O}_E^{\mathbb{C}} := \mathcal{O}_{\mathcal{L}(E)}^{\mathcal{R}_{\mathbb{C}}}).$$

We shall show in our cases of interest, both orbits are so-called *formal submanifolds*.

Formal submanifolds and manifold ideals

It is a consquence of a "formal variant of Nagano's theorem" by Baouendi-Ebenfelt-Rothschild (an adaptation to the formal case of my unpublished proof of Nagano's theorem), that the real formal orbit $\mathcal{O}_E^{\mathbb{R}}$ is always a so-called *real formal submanifold*.

The definition is an adaptation of the definition of a germ at p of a smooth submanifold $M \subset \mathbb{C}^n$, whose ideal $I(M)$ in the ring of smooth function germs at p is generated by a finite subset with linearly independent differentials at p:

Definition

1. A manifold ideal I in a ring $\mathcal{R} \subset \mathbb{C}[[z-p, \overline{z-p}]]$ is one generated by k elements $F_1, \ldots, F_k \in I$ with linearly independent differentials at p.
2. A *real (resp. complex) formal submanifold* O at p in $\mathbb{C}^n$ is one defined by a *manifold ideal* $I = I(O)$ in $\mathcal{R}_{\mathbb{R}}$ (resp. $\mathcal{R}_{\mathbb{C}}$).

Note that, unlike germs of smooth functions, we can only evaluate differentials of formal power series at the base point p.

Orbits of subbundles of infinite Levi type

A small abuse of notation

We assume the contact form θ on S is extended to a neighborhood of p in $\mathbb{C}^n$ (our construction will be independent of this extension).

Notation for formal Taylor series of smooth functions

Given a smooth function f in a neighborhood of p in $\mathbb{C}^n$, we write $j_p^\infty f \in \mathbb{C}[[z-p, \overline{z-p}]]$ for its *formal Taylor series* at p.

As direct consequence of our definition of the real orbit, we obtain:

Corollary

If E has infinite Levi type $c(E,p)=\infty$, i.e. $L^m \cdots L^3\theta([L^2,L^1])(p)=0$ for all $L^j \in \Gamma(E)\cup\Gamma(\overline{E})$, then

$$j_p^\infty \theta([L^2,L^1]) \in \mathbb{C}I(\mathcal{O}_E^{\mathbb{R}})$$

for all complex vector fields $L^2, L^1 \in \mathcal{L}(E)$.

Weighted expansion of nonnegative functions

Our next goal is to use *pseudoconvexity* to obtain some stronger conclusions. We need the following well-known lemma:

Lemma (positivity of the lowest weight component)

Let $f \geq 0$ be a nonnegative smooth function in a neighborhood of 0 in $\mathbb{R}^m$. Fix a collection of positive weights $\mu_j > 0$, $j = 1, \ldots, m$, and number $k \geq 0$, and consider a decomposition

$$f = f_k + f_{>k},$$

where f_k is weighted homogeneous of degree k, while the Taylor expansion of $f_{>k}$ at 0 consists of terms of weight $> k$. Then $f_k \geq 0$.

Proof The conclusion is obtained by taking the limit of

$$t^{-k} f(t^{\mu_1} x_1, \ldots, t^{\mu_m} x_m) \geq 0$$

as $t \to 0$ for each fixed $(x_1, \ldots, x_m) \in \mathbb{R}^m$.

Formal parametrizations of formal submanifolds

By the *implicit function theorem for formal power series*, after possible reordering coordinates, a manifold ideal I with *k real generators* also admits generators of the form

$$x_{j+m-k} - \phi_j(x'), \quad x' = (x_1, \ldots, x_{m-k}), \quad j = 1, \ldots, k, \tag{1}$$

and hence the real formal power series map

$$A(x') := (x', \phi_1(x'), \ldots, \phi_k(x'))$$

satisfies

$$\operatorname{rank} dA(0) = k, \quad F \circ A = 0 \text{ for all } F \in I.$$

Definition

Given a formal submanifold O with $I(O)$ having k real generators, a real formal power series map H with

$$\operatorname{rank} dH(0) = k, \quad F \circ A = 0 \text{ for all } F \in I(O).$$

is called a (formal) parametrization of O.

Pseudoconvexity $\Longrightarrow$ Levi null cone coincides with kernel

Notation

1. Write $j_p^\infty S$ for the formal submanifold defined by the ideal of all Taylor series of germs of smooth functions vanishing on S.
2. Write $O \subset O'$ for formal submanifolds O, O' whenever $I(O) \supset I(O')$.

In particular, for a subbundle $E \subset \mathbb{C}TM$, one has $\mathcal{O}_E^{\mathbb{R}} \subset j_0^\infty S$.
A well-known result from Linear Algebra for a *positive semi-definite* hermitian form $h(\cdot, \cdot)$ asserts that $h(L, L) = 0$ for some L implies $h(L, L') = 0$ for any L'. We show a formal variant of this statement:

Proposition (Levi null cone vs Levi kernel)

Let $S \subset \mathbb{C}^n$ be *pseudoconvex*, $0 \in S$, and $O \subset j_0^\infty S$ a real formal submanifold. Let L, L' be smooth complex vector fields in a neighborhood of 0 in $\mathbb{C}^n$ with $L|_S, L'|_S \in H^{10}S$. Then

$$j_0^\infty \theta([L, \overline{L}]) \in \mathbb{C}I(O) \implies j_0^\infty \theta([L, \overline{L'}]) \in \mathbb{C}I(O).$$

Proof of the Levi cone vs kernel proposition

Pseudoconvexity with suitable choice of θ implies for all $c \in \mathbb{C}$:

$$\theta([L + cL', \overline{L + cL'}]) = \theta([L, \overline{L}]) + 2\mathrm{Re}(\overline{c}\theta([L, \overline{L'}])) + c\overline{c}\theta([L', \overline{L'}]) \geq 0$$

Since $O \subset j_p^\infty S$, there exists a smooth map germ $\gamma\colon (\mathbb{R}^q, 0) \to (S, 0)$ such that $j_0^\infty \gamma$ is a formal parametrization of O. Take pullbacks under γ:

$$\gamma^*\theta([L, \overline{L}]) + 2\mathrm{Re}(\overline{c}\gamma^*\theta([L, \overline{L'}])) + c\overline{c}\gamma^*\theta([L', \overline{L'}]) \geq 0'$$

Since $j_0^\infty \theta([L, \overline{L'}]) \in \mathbb{C}I(O) \iff \gamma^*\theta([L, \overline{L'}])$ vanishes of infininte order, *assume by contradiction that order s to be finite*. Assign weights 1 to $(t_1, \ldots, t_q) \in \mathbb{R}^q$ and $\mu = s + 1$ to $(\mathrm{Re}c, \mathrm{Im}c)$. The middle term has the lowest weight term $2\mathrm{Re}(\overline{c}\gamma^*\theta([L, \overline{L'}]))_{s+\mu} \neq 0$, $c\overline{c}\gamma^*\theta([L', \overline{L'}])$ has weight $\geq 2\mu > s + \mu$, and $\gamma^*\theta([L, \overline{L}])$ vanishes of infinite order when $j_p^\infty \theta([L, \overline{L}]) \in \mathbb{C}I(O)$. Therefore, the lowest weight term satisfies

$$2\mathrm{Re}(\overline{c}\gamma^*\theta([L, \overline{L'}]))_{s+\mu} \geq 0$$

by positivity lemma. Since c is arbitrary, the left-hand side must vanish, which contradicts the above conclusion in red, completing the proof. □

Complex-tangential property for real orbits

Notation

1. For a subbundle E, denote by $j_p^\infty E$ the space of all *formal Taylor expansions of complex vector fields* in whose restrictions to S are in E.
2. For a formal submanifold O, denote by D_O the module of all *formal vector fields* L tangent to O in the sense of $L(I(O)) \subset I(O)$.

Corollary

Let S be pseudoconvex and a subbundle $E \subset H^{10}S$ be of infinite Levi type *at $p \in S$. Then $O = O_E^{\mathbb{R}}(p)$ satisfies:*

1. $[j_p^\infty(E \oplus \overline{E}), j_p^\infty \mathbb{C}HS] \subset j_p^\infty \mathbb{C}HS \mod \mathbb{C}I(O)$.
2. $[D_O, j_p^\infty \mathbb{C}HS] \subset j_p^\infty \mathbb{C}HS \mod \mathbb{C}I(O)$.
3. *The orbit O is* complex-tangential *to $j_p^\infty S$ in the sense that*

$$D_O \subset j_p^\infty \mathbb{C}HS \mod \mathbb{C}I(O).$$

Proof of (1)

Consider formal vector fields in $j_p^\infty E$ and $j_p^\infty H^{10}S$ which are Taylor series at p of smooth vector fields L, L' whose restrictions to S are in E and $H^{10}S$ respectively. Since E is of *infinite Levi type*,

$$j_p^\infty \theta([L, \overline{L}]) \in \mathbb{C}I(\mathcal{O}),$$

and the "Levi null cone vs kernel proposition" implies

$$j_p^\infty \theta([L, \overline{L'}]) \in \mathbb{C}I(O).$$

The above holds also for L' with $L'|_S \in H^{01}S$, since $H^{10}S$ is an involutive distribution in $\mathbb{C}TS$ by the integrability of the CR structure. Since θ is a contact form,

$$[j_p^\infty E, j_p^\infty \mathbb{C}HS] \subset j_p^\infty \mathbb{C}HS \mod \mathbb{C}I(O),$$

from which (1) follows, since the right-hand side is invariant under conjugation.

Proof of (2) and (3)

To show (2), observe that (1) along with Jacobi identity implies that

$$[L, j_p^\infty \mathbb{C}HS] \subset j_p^\infty \mathbb{C}HS \mod \mathbb{C}I(O)$$

for any L which is an iterated Lie bracket of formal vector fields in $j_p^\infty(E \oplus \overline{E})$. Since $O = \mathcal{O}_E^{\mathbb{R}}(p)$ is the orbit of the Lie algebra $\mathfrak{g} = \mathfrak{g}_E^{\mathbb{R}}$ spanned by the real parts $\mathrm{Re}L = \frac{1}{2}(L + \overline{L})$ of iterated Lie brackets, it follows that

$$[\mathfrak{g}, j_p^\infty \mathbb{C}HS] \subset j_p^\infty \mathbb{C}HS \mod \mathbb{C}I(O).$$

It can be shown that the orbit O satisfies $D_O = \mathfrak{g} \mod I(O)$, proving (2).

Now, since $E \oplus \overline{E} \subset \mathbb{C}HS$, by repeatedly using (1), we conclude that any iterated Lie bracket of vector fields in $j_p^\infty(E \oplus \overline{E})$ is contained in $j_p^\infty \mathbb{C}HS$ modulo $\mathbb{C}I(O)$. As before, using the relation $D_O = \mathfrak{g} \mod I(O)$ completes the proof of (3).

Finite commutator type implies finite tower multitype

We use the above Corollary, part (3) to conclude the inclusion of tangent spaces

$$T_p O \subset H_p S,$$

where the $T_p O$ is defined by the vanishing of all $df(p)$ with $f \in I(O)$. We now restate and prove the main theorem of this lecture:

Theorem (finite commutator type only for special subbundles)

Assume $t(E, p) < \infty$ for any special subbundle $E \subset H^{10}S$ of rank 1. Then S is of finite tower multitype at p.

Proof. It suffices to show finite Levi type at p for any *special subbundle* $E \subset H^{10}S$ of rank ≥ 1, since that implies *finite tower multitype* at p as we proved in Lecture 1. Assume by contradiction, there is a special subbundle E of rank ≥ 1, which is of *infinite Levi type* at p. Then the above inclusion of tangent spaces holds, and since formal Taylor series of any iterated commutator $[L^m, \dots, [L^2, L^1] \dots]$ for $L^m, \dots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E})$, are tangent to the formal orbit $O = \mathcal{O}_E^{\mathbb{R}}(p)$, their values at p stay in $\mathbb{C}H_p S$, contradicting $t(E, p) < \infty$.