New tools and conditions for global regularity of the $\bar{\partial}$-Neumann operator

Part 1 - Tower Multitype

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Notation: local defining functions

1. *smooth* means always C^∞;
2. $S \subset \mathbb{C}^{n+1}$, $n \geq 1$, (or $S \subset \mathbb{C}^n$, $n \geq 2$) is a smooth real hypersurface;
3. a *local defining function* r of S in a neighborhood U of $p \in S$ is any smooth real function with

   $$S \cap U = \{r = 0\}$$

   and $dr \neq 0$ at every point of U;
4. any two local defining functions r_1, r_2 in U differ by a *nonzero smooth function factor*:

   $$r_2(x) = a(x) r_1(x), \quad a(x) \neq 0.$$

Notation: tangent bundles

1. TS is the *real tangent bundle*;
2. $\mathbb{C}TS = \mathbb{C} \otimes_{\mathbb{R}} TS$ is the *complexified tangent bundle*;
3. $H^{10}S = \{X \in \mathbb{C}TS : \partial r(X) = 0\}$ is the $(1,0)$ bundle;
4. $H^{01}S = \{X \in \mathbb{C}TS : \bar{\partial} r(X) = 0\}$ is the $(0,1)$ bundle;
5. $HS = \mathrm{Re} H^{10}S = \mathrm{Re} H^{01}S \subset TS$ is the *complex tangent bundle*;
6. We have the standard relations:

$$H^{01}S = \overline{H^{10}S}, \quad \mathbb{C}HS = H^{10}S \oplus \overline{H^{10}S}.$$

Motivation: from finiteness conditions to global regularity

finiteness conditions $\Longrightarrow$ generalized stratification

generalized stratification $\Downarrow$ property (P_q)

property (P_q) $\Downarrow$ compactness

global regularity $\Longleftarrow$ compactness

global regularity $\Uparrow$ condition R

Examples of finiteness conditions and relations between them:

finite D'Angelo type $\Longrightarrow$ finite regular type

finite regular type $\Downarrow$ finite tower multitype

Here we shall focus on the more general *finite tower multitype* condition and the first implication shown in red.

Generalized stratifications with convexity properties

The goal in the first implication is to obtain generalized stratifications of the hypersurface S with certain *convexity properties*:

Definition (generalizing regular domains by Catlin-Diederich-Fornaess)

A hypersurface $S \subset \mathbb{C}^{n+1}$ is countably q-regular ($1 \le q \le n$) if it is a countable disjoint union $S = \cup_{k=1}^{\infty} S_k$ of locally closed subsets $S_k \subset S$ ("strata") such that for each k and $p \in S_k$, there exists a CR submanifold $M \subset S$ satisfying the following properties:

1. M contains an open neighborhood of p in S_k (in relative topology);
2. $\dim_{\mathbb{C}}(H_x^{10} M \cap K_x^{10}) < q$ for all $x \in M$, where $K_x^{10} \subset H_x^{10} S$ is the *kernel of the Levi form* of S.

When $q = 1$ and S is pseudoconvex, condition (2) simply means that the Levi form of S is *positive definite along $H^{10} M$*. This allows constructions of *bounded* local *weight functions aka barriers aka bumps* with *large complex hessians* on strata S_k as $C(r + \sum_j r_j^2)$, where r (resp. r_j) are local defining functions of S (resp. M).

Bounded barriers with large complex hessians (BBLH)

The *complex hessian* of a real function λ is the hermitian quadratic form

$$H_\lambda(X) := \sum \lambda_{z_j \bar{z}_k} X_j \overline{X}_k.$$

By definition, property $(P_1) \iff$ the existence of BBLH on a set A means the existence of functions λ with $0 \le \lambda \le 1$ in a neighborhood of A with arbitrarily large complex hessian (the neighborhood of A depends on how large is the hessian).

Sibony's B-regularity theory

Local existence of BBLH for strata S_k implies global existence of BBLH for their countable unions.

Vast applications of BBLH — *passing from flexible to rigid objects*:

1. a priori estimates for $\bar\partial$ leading to compactness and global regularity by Kohn-Nirenberg;
2. regularity of the Bergman projection aka *condition R* by Bell-Ligocka implying boundary smoothness of proper holomorphic maps;
3. estimates for reproducing kernels and invariant metrics.

Back to our tools: Forms dual to lists of vector fields

Definition

1. A *complex contact form* θ on S is any nonzero $\mathbb{C}$-valued 1-form vanishing on $\mathbb{C}HS = H^{10}S \oplus \overline{H^{10}S} \subset \mathbb{C}TS$.
2. The *θ-dual form* of an (ordered) list of complex vector fields

$$L^t, \dots, L^1 \in \Gamma(H^{10}S) \cup \Gamma(\overline{H^{10}S})), \quad t \geq 1,$$

is the complex 1-form $\omega_{L^t,\dots,L^1;\theta}$ on $H^{10}S$ defined for $L \in \Gamma(H^{10}S)$, $p \in S$, by

$$\begin{cases} \omega_{L^1;\theta}(L_p) := \theta([L, L^1])(p) & t = 1 \\ \omega_{L^t,\dots,L^1;\theta}(L_p) := L\mathrm{Re}(L^t \cdots L^3\theta([L^2, L^1]))(p), & t \geq 2 \end{cases}.$$

A complex contact form is defined up to a nonzero smooth function factor. If θ is purely imaginary on S, e.g. $\theta = \partial r$, then the *Levi form* of S is

$$\mathrm{Levi}_S(L) = \theta([L, \overline{L}]).$$

Towers on real hypersurfaces

Let $S \subset \mathbb{C}^{n+1}$ be a smooth real hypersurface, θ a complex contact form.

1. A complex 1-form ω defined on $H^{10}S$ is called E-dual of order $t \in \mathbb{N}_{\geq 2}$, where $E \subset H^{10}S$ is a complex subbundle, if it is θ-dual of a list of $(t-1)$ complex vector fields

$$L^{t-1}, \ldots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E})).$$

2. A tower on S of multi-order $(t_1, \ldots, t_n) \in (\mathbb{N}_{\geq 2} \cup \{\infty\})^n$ is a nested sequence of complex subbundles

$$H^{10}S = E_0 \supset \ldots \supset E_m, \quad 0 \leq m \leq n,$$

such that $t_{m+1} = \ldots = t_n = \infty$, and for each $k = 1, \ldots, m$, one has $t_k \in \mathbb{N}_{\geq 2}$ and there exists an E_{k-1}-dual form ω_k of order t_k with

$$E_k = E_{k-1} \cap \{\omega_k = 0\}, \quad \omega_k|_{E_{k-1}} \neq 0.$$

Functions associated with towers

1. The θ-dual form of the list of $t \geq 2$ vector fields $L^t, \ldots, L^1 \in \Gamma(H^{10}S) \cup \Gamma(\overline{H^{10}S})$ can be written as

$$\omega_{L^t,\ldots,L^1;\theta}(L_p) := Lf_{L^t,\ldots,L^1;\theta}(p), \quad f_{L^t,\ldots,L^1;\theta} := \mathrm{Re}(L^t \cdots L^3\theta([L^2, L^1])),$$

   where we call $f_{L^t,\ldots,L^1;\theta}$ the θ-dual function of $(L^t, \ldots, L^1)$.

2. For any tower

$$H^{10} = E_0 \supset \ldots \supset E_m, \quad 0 \leq m \leq n, \quad E_k = E_{k-1} \cap \{\omega_k = 0\},$$

   and any choice of vector fields (L^s_k) with $\omega_k = \omega_{L_k^{t_k-1},\ldots,L_k^1;\theta}$, collect all θ-dual functions *for all k with $t_k \geq 2$* into the set

$$\{f_{L_k^{t_k-1},\ldots,L_k^1;\theta} : t_k \geq 2\}$$

   that we call an associated set of functions of the given tower.

First structure property

Proposition (used to obtain convexity properties of stratifications)

Let $S \subset \mathbb{C}^{n+1}$ be a smooth real hypersurface and

$$H^{10}S = E_0 \supset \ldots \supset E_m,$$

a tower of the multi-order $(t_1, \ldots, t_n)$ on S with an *associated set of functions* $\{f_1, \ldots, f_l\}$ $(l \leq m)$. Then the following hold:

1. The restrictions to $H^{10}S$ of the differentials $df_1, \ldots, df_l$ are linearly independent, in particular, the zero set

$$M := \{f_1 = \ldots = f_l = 0\} \subset S$$

   is a smooth *CR submanifold*.

2. The kernel (nullspace) distribution (of varying rank) $K^{10} \subset H^{10}S$ of the Levi form of S satisfies

$$H^{10}M \cap K^{10} \subset E_m.$$

Proof of the first structure property

1. It follows from the definition of a tower that the forms $\omega_1, \ldots, \omega_m$ defined there are linearly independent when restricted to $H^{10}S$. If the set $\{f_1, \ldots, f_l\}$ is empty, (1) is void. Otherwise, the equality of the sets of the forms

$$\{\omega_k : t_k \geq 2\} = \{df_j|_{H^{10}S} : 1 \leq j \leq l\}$$

proves (1).

2. To show (2), let $\xi \in K^{10}$. Since $\omega_k = \omega_{L_k;\theta} = \theta([\cdot, L_k])$ when $t_k = 1$, it follows that $\omega_k(\xi) = 0$. On the other hand, when $t_k \geq 2$, $\xi \in H^{10}M$ implies $\omega_k(\xi) = df_{l_k}(\xi) = 0$ for some $l_k \in \{1, \ldots, l\}$. Hence

$$\xi \in K^{10} \cap H^{10}M \implies \xi \in H^{10}S \cap \{\omega_1 = \ldots = \omega_m = 0\} = E_m$$

as desired.

Tower multitype: definition

The strata of S are obtained as level sets of the multitype function:

Definition

The tower multitype of S at $p \in S$ is the CR-invariant

$$\mathcal{T}(p) \in (\mathbb{N}_{\geq 2} \cup \{\infty\})^n$$

defined as the lexicographically minimum multi-order $(t_1, \dots, t_n)$ of a tower on a neighborhood of p in S.

Here the *lexicographic order* is defined in the standard way by

$$(t_1, \dots, t_n) < (t'_1, \dots, t'_n) \iff \exists\, k \in \{1, \dots, n-1\}, (t_1, \dots, t_{k-1}) = (t'_1, \dots, t'_{k-1}),\ t_k < t'_k.$$

Taking the lexicographic order in (3) guarantees that $\mathcal{T}(p)$ is an invariant only depending on the CR structure of S (in fact, only on the Levi form).

Tower multitype: independence of the contact form

Recall: θ-dual forms $\omega_{L^t,\dots,L^1;\theta}$ are defined by

$$\begin{cases} \omega_{L^1;\theta}(L_p) := \theta([L, L^1])(p) & t = 1 \\ \omega_{L^t,\dots,L^1;\theta}(L_p) := L\mathrm{Re}(L^t \cdots L^3\theta([L^2, L^1]))(p), & t \geq 2 \end{cases}.$$

Independence of θ

Any other complex contact form satisfies $\widetilde{\theta} = h\theta$, where h is a nonzero smooth complex function. Then

$$\omega_{L_k^{t_k-1},\dots,L_k^1;h\theta} = \omega_{L_k^{t_k-1},\dots,hL_k^1;\theta}$$

and with $\widetilde{\theta}$ instead of θ, we can modify the vector fields (L_k^s) to obtain the same forms ω_k defining the same tower.

Second structure property

Level sets of $\mathcal{T}$ serve as generalized strata, whose properties follow from

Proposition (used for local inclusion of strata into submanifolds)

Let $S \subset \mathbb{C}^{n+1}$ be a smooth real hypersurface, $p \in S$ a point, $U \subset S$ an open neighborhood of p, and

$$H^{10}U = E_0 \supset \ldots \supset E_m$$

a tower on U, whose multi-order equals the multitype $\mathcal{T}(p)$. Choose any associated set of functions

$$\{f_1, \ldots, f_l\}.$$

Then the following hold:

1. $\mathcal{T}(p') \le \mathcal{T}(p)$ for any $p' \in U$ (with respect to lexicographic order);
2. the tower multitype level set satisfies

$$\{p' \in U : \mathcal{T}(p') = \mathcal{T}(p)\} \subset \{f_1 = \ldots = f_l = 0\}.$$

Proof of the second structure property

Since $\mathcal{T}(p) = (t_1, \ldots, t_n)$ is the multi-order of the given tower on U and $\mathcal{T}(p')$ is the minimum multi-order for a tower on a neighborhood of p', (1) is immediate.

To show (2), choose $p' \in U$ with $f_j(p') \neq 0$ for some $j = 1, \ldots, l$, where

$$f_j(p') = \mathrm{Re}(L_k^{t_k-1} \cdots L_k^3 \theta([L_k^2, L_k^1]))(p') \neq 0, \quad L_k^{t_k-1}, \cdots, L_k^1 \in E_{k-1} \cup \overline{E}_{k-1}$$

for some k that we choose to be *minimal with this property* for any j. If $L_k^{t_k-1} \in \overline{E}_{k-1}$, taking conjugates of all vector fields and of θ and replacing θ with $f\theta$, where f is a nonzero function, we may assume that $L_k^{t_k-1} \in E_{k-1}$. In particular, we obtain

$$\omega_k'|_{(E_{k-1})_{p'}} \neq 0,$$

where for $x \in S$, $L \in \Gamma(H^{10}S)$,

$$\omega_k'(L_x) := \theta([L, L_k^1)(x) \text{ for } t_k = 3,$$

or

$$\omega_k'(L_x) := L L_k^{t_k-2} \cdots L_k^3 \theta([L_k^2, L_k^1]))(x) \text{ for } t_k > 3.$$

Proof of the second structure property, part (2), continued

In the case $t_k = 3$, we have $\omega'_k = \omega_{L_k^1;\theta}$. In the case $t_k > 3$, splitting into real and imaginary parts, we obtain

$$\omega'_k(L_{p'}) = L(\mathrm{Re} f + i \mathrm{Im} f)(p'), \quad f := L_k^{t_k-2} \cdots L_k^3 \theta([L_k^2, L_k^1])).$$

Taking the term that does not identically vanish for $L_{p'} \in (E_{k-1})_{p'}$ and multiplying $L_k^{t_k-2}$ by i if necessary, we may assume $\omega'_k(L_x) = L\mathrm{Re} f(x)$, hence $\omega'_k = \omega_{L_k^{t_k-2},\ldots,L_k^1;\theta}$. In both cases, we obtain a *new tower*

$$H^{10}U' = E_0 \supset \ldots \supset E_{k-1} \supset E'_k$$

in a neighborhood $U' \subset U$ of p' of the *lexicographically smaller multi-order*

$$(t_1, \ldots, t_{k-1}, t_k - 1, \infty, \ldots, \infty) < (t_1, \ldots, t_{k-1}, t_k, \ldots, t_n),$$

by setting

$$E'_k := E_{k-1} \cap \{\omega'_k = 0\}.$$

By definition of the tower multitype, $\mathcal{T}(p')$ is the minimum multi-order for a tower in its neighborhood, hence $\mathcal{T}(p') < \mathcal{T}(p)$ and thus p' is not in the level set of p, completing the proof of (2).

Consequences of the second structure property

Since the tower multitype only takes discrete values

$$\mathcal{T}(p) \in (\mathbb{N}_{\geq 2} \cup \{\infty\})^n, \quad p \in S,$$

part (1) of the second structure property immediately yields:

Corollary

For a smooth real hypersurface $S \subset \mathbb{C}^{n+1}$, the following hold:

1. *The tower multitype function $\mathcal{T}$ is* upper-semicontinuous.
2. *Level sets of $\mathcal{T}$ are* locally closed, *i.e. closed in their open neighborhoods.*

Recall:

Our goal is to obtain a generalized stratification with convexity propreties using level sets of $\mathcal{T}$ as strata:

$$S = \bigcup_{(t_1,\ldots,t_n)\in(\mathbb{N}_{\geq 2}\cup\{\infty\})^n} \{p : \mathcal{T}(p) = (t_1, \ldots, t_n)\}.$$

Hypersurfaces of finite tower multitype

For simplicity, we shall consider the case $q = 1$.

Theorem (generalized stratification for finite tower multitype)

Let $S \subset \mathbb{C}^{n+1}$ be a (not necessarily pseudoconvex) smooth hypersurface whose tower multitype has all entries finite at every point. Then S is countably 1-regular, where the "strata" can be chosen to be the level sets of the tower multitype function $\mathcal{T}$.

Recall:

A hypersurface $S \subset \mathbb{C}^{n+1}$ is countably 1-regular if it is a countable disjoint union $S = \cup_{k=1}^{\infty} S_k$ of locally closed subsets $S_k \subset S$ ("strata") such that for each k and $p \in S_k$, there exists a CR submanifold $M \subset S$ satisfying the following properties:

1. M contains an open neighborhood of p in S_k (in relative topology);
2. $H_x^{10} M \cap K_x^{10} = \{0\}$ for all $x \in M$, where $K_x^{10} \subset H_x^{10} S$ is the *kernel of the Levi form* of S.

Proof of countable 1-regularity for finite tower multitype

Since the tower multitype of S is finite at every point, S splits into the countable disjoint union of the $\mathcal{T}$-level sets

$$S = \bigcup_{(t_1,\dots,t_n)\in\mathbb{N}^n_{\geq 2}} \{p : \mathcal{T}(p) = (t_1, \dots, t_n)\}.$$

By the above corollary, each level set of $\mathcal{T}$ is locally closed, and by the second structure property, it is locally contained in the zero set

$$M = \{f_1 = \dots = f_l = 0\},$$

where $\{f_1, \dots, f_l\}$ is an *associated set of functions* of a tower

$$H^{10}S = E_0 \supset \dots \supset E_m$$

on an open subset of S.

Since all entries of $\mathcal{T}$ are finite, E_m is the zero subbundle. Then by the first structure property, M is a CR submanifold of S satisfying

$$H^{10}M \cap K^{10} \subset E_m = 0,$$

which is precisely the desired convexity property.

Hypersurfaces with subbundles of finite Levi type

We have shown the implication

$$\text{finite tower multitype} \implies \text{countable 1-regularity},$$

from which known results imply *compactness* and *global regularity*. A simpler assumption going back to Kohn's 1972 JDG paper, is based on

Definition

The Levi type $c(E,p) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ at $p \in S$ of a subbundle $E \subset H^{10}S$ is

$$\min\{t \geq 2 : \exists L^t, \ldots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E}), L^m \cdots L^3 \partial r([L^2, L^1])(p) \neq 0\},$$

where r is a local defining function of S.

Corollary

For a (not necessarily pseudoconvex) smooth hypersurface $S \subset \mathbb{C}^{n+1}$*, assume* $c(E,p) < \infty$ *for any smooth subbundle* $E \subset H^{10}S$ *of rank* 1 *and any* $p \in S$*. Then* S *is countably* 1*-regular.*

Proof of the corollary

It suffices to reduce to the case of *finite tower multitype* treated above. Assume by contradiction that for some for some $p \in S$, $\mathcal{T}(p)$ is not finite, i.e. *some of the entries are infinite*. By definition, the tower multitype at p is realized as the multi-order of a tower in a neighborhood U of p

$$H^{10}U = E_0 \supset \ldots \supset E_m.$$

Since not all entries finite, $E_m \neq \{0\}$.
We claim that $c(E, p) = \infty$, which will contradict our assumption, hence completing the proof. Indeed, otherwise

$$L^t \cdots L^3 \theta([L^2, L^1])(p) \neq 0$$

for some $t \geq 2$ and some choice of vector fields $L^t, \ldots, L^1 \in E_m \cup \overline{E}_m$. Then, by repeating the arguments of the proof of the second structure property, we reach a contradiction with the tower multitype definition constructing another tower on a neighborhood of p in S of a lexicographically smaller multi-order

$$(t_1, \ldots, t_m, t_{m+1}, \ldots, t_n) < (t_1, \ldots, t_m, \infty, \ldots, \infty).$$

Special subbundles

Our argument yields in fact a stronger more refined version of the above corollary, where the *Levi type* finiteness $c(E,p) < \infty$ only needs to be checked for certain *special subbundles* E that always arise in a tower:

Definition (special subbundle)

A complex subbundle $E \subset H^{10}S$ is called special if it can be defined by

$$E = \{\xi \in H^{10}S : \omega_1(\xi) = \ldots = \omega_l(\xi) = 0\}, \quad \omega_1 \wedge \cdots \wedge \omega_l \neq 0 \text{ on } (H^{10}S)^l,$$

where each ω_j, $j = 1, \ldots, l$, is the *θ-dual 1-form* $\omega_j = \omega_{L_j^{t_j},\ldots,L_j^1}$ for some $t_j \geq 1$ and vector fields $L_j^{t_j}, \ldots, L_j^1 \in \Gamma(H^{10}S) \cup \Gamma(\overline{H^{10}S})$.

Theorem (finite Levi type only for special subbundles)

Assume $c(E,p) < \infty$ for any special subbundle *$E \subset H^{10}S$ of rank ≥ 1. Then S is of* finite tower multitype *at p.*

Note that $c(E,p) < \infty$ implies $c(E',p) < \infty$ for any subbundle $E' \subset E$.