New tools and conditions for global regularity of the $\bar{\partial}$-Neumann operator

Part 1 - Tower Multitype

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Notation: local defining functions

1. *smooth* means always $C^\infty$;  
2. $S \subset \mathbb{C}^{n+1}$, $n \geq 1$, (or $S \subset \mathbb{C}^n$, $n \geq 2$) is a smooth real hypersurface;  
3. a *local defining function* $r$ of $S$ in a neighborhood $U$ of $p \in S$ is any smooth real function with  
   
   $$S \cap U = \{ r = 0 \}$$  

   and $dr \neq 0$ at every point of $U$;  
4. any two local defining functions $r_1, r_2$ in $U$ differ by a *nonzero smooth function factor*:  
   
   $$r_2(x) = a(x)r_1(x), \quad a(x) \neq 0.$$
1. $TS$ is the \textit{real tangent bundle};
2. $\mathbb{C}TS = \mathbb{C} \otimes_{\mathbb{R}} TS$ is the \textit{complexified tangent bundle};
3. $H^{10} S = \{ X \in \mathbb{C}TS : \partial r(X) = 0 \}$ is the $(1, 0)$ bundle;
4. $H^{01} S = \{ X \in \mathbb{C}TS : \bar{\partial} r(X) = 0 \}$ is the $(0, 1)$ bundle;
5. $HS = \text{Re}H^{10} S = \text{Re}H^{01} S \subset TS$ is the \textit{complex tangent bundle};
6. We have the standard relations:

\[
H^{01} S = \overline{H^{10} S}, \quad \mathbb{C}HS = H^{10} S \oplus \overline{H^{10} S}.
\]
From finiteness conditions to global regularity

Examples of finiteness conditions and relations between them:

- finite D’Angelo type $\rightarrow$ finite regular type $\rightarrow$ finite tower multitype

Here we shall focus on the more general \textit{finite tower multitype} condition and the first implication shown in red.
The goal in the first implication is to obtain generalized stratifications of the hypersurface $S$ with certain \textit{convexity properties}:

\textbf{Definition (generalizing regular domains by Catlin-Diederich-Fornaess)}

A hypersurface $S \subset \mathbb{C}^{n+1}$ is \textit{countably $q$-regular} ($1 \leq q \leq n$) if it is a countable disjoint union $S = \bigcup_{k=1}^{\infty} S_k$ of locally closed subsets $S_k \subset S$ ("strata") such that for each $k$ and $p \in S_k$, there exists a CR submanifold $M \subset S$ satisfying the following properties:

1. $M$ contains an open neighborhood of $p$ in $S_k$ (in relative topology);
2. $\dim_{\mathbb{C}} (H_x^{10} M \cap K_x^{10}) < q$ for all $x \in M$, where $K_x^{10} \subset H_x^{10} S$ is the \textit{kernel of the Levi form} of $S$.

When $q = 1$ and $S$ is pseudoconvex, condition (2) simply means that the Levi form of $S$ is \textit{positive definite along $H^{10} M$}. This allows constructions of \textit{bounded local weight functions aka barriers aka bumps} with \textit{large complex hessians} on strata $S_k$ as $C(r + \sum_j r_j^2)$, where $r$ (resp. $r_j$) are local defining functions of $S$ (resp. $M$).
The **complex hessian** of a real function $\lambda$ is the hermitian quadratic form

$$H_\lambda(X) := \sum \lambda_{z_j\bar{z}_k} X_j \bar{X}_k.$$ 

By definition, property $(P_1) \iff$ the existence of BBLH on a set $A$ means the existence of functions $\lambda$ with $0 \leq \lambda \leq 1$ in a neighborhood of $A$ with arbitrarily large complex hessian (the neighborhood of $A$ depends on how large is the hessian).

**Sibony’s $B$-regularity theory**

Local existence of BBLH for strata $S_k$ implies global existence of BBLH for their countable unions.

Vast applications of BBLH — *passing from flexible to rigid objects*:

1. A priori estimates for $\bar{\partial}$ leading to compactness and global regularity by Kohn-Nirenberg;
2. Regularity of the Bergman projection aka *condition R* by Bell-Ligocka implying boundary smoothness of proper holomorphic maps;
3. Estimates for reproducing kernels and invariant metrics.
A complex contact form $\theta$ on $S$ is any nonzero $\mathbb{C}$-valued 1-form vanishing on $\mathbb{C}HS = H^{10}S \oplus \overline{H^{10}S} \subset \mathbb{C}TS$.

The $\theta$-dual form of an (ordered) list of complex vector fields $L^t, \ldots, L^1 \in \Gamma(H^{10}S) \cup \Gamma(\overline{H^{10}S})$, $t \geq 1$, is the complex 1-form $\omega_{L^t, \ldots, L^1; \theta}$ on $H^{10}S$ defined for $L \in \Gamma(H^{10}S)$, $p \in S$, by

$$
\begin{align*}
\omega_{L^1; \theta}(L_p) &:= \theta([L, L^1])(p) & t = 1 \\
\omega_{L^t, \ldots, L^1; \theta}(L_p) &:= L\text{Re}(L^t \cdots L^3 \theta([L^2, L^1]))(p), & t \geq 2.
\end{align*}
$$

A complex contact form is defined up to a nonzero smooth function factor. If $\theta$ is purely imaginary on $S$, e.g. $\theta = \partial r$, then the Levi form of $S$ is

$$\text{Levi}_S(L) = \theta([L, \overline{L}]).$$
Towers on real hypersurfaces

Let $S \subset \mathbb{C}^{n+1}$ be a smooth real hypersurface, $\theta$ a complex contact form.

1. A complex 1-form $\omega$ defined on $H^{10}S$ is called $E$-dual of order $t \in \mathbb{N}_{\geq 2}$, where $E \subset H^{10}S$ is a complex subbundle, if it is $\theta$-dual of a list of $(t-1)$ complex vector fields

$$L^{t-1}, \ldots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E}).$$

2. A tower on $S$ of multi-order $(t_1, \ldots, t_n) \in (\mathbb{N}_{\geq 2} \cup \{\infty\})^n$ is a nested sequence of complex subbundles

$$H^{10}S = E_0 \supset \ldots \supset E_m, \quad 0 \leq m \leq n,$$

such that $t_{m+1} = \ldots = t_n = \infty$, and for each $k = 1, \ldots, m$, one has $t_k \in \mathbb{N}_{\geq 2}$ and there exists an $E_{k-1}$-dual form $\omega_k$ of order $t_k$ with

$$E_k = E_{k-1} \cap \{\omega_k = 0\}, \quad \omega_k|_{E_{k-1}} \neq 0.$$
Functions associated with towers

1. The $\theta$-dual form of the list of $t \geq 2$ vector fields $L^t, \ldots, L^1 \in \Gamma(H^{10}S) \cup \Gamma(H^{10}S^\perp)$ can be written as

$$\omega_{L^t,\ldots,L^1;\theta}(L_p) := Lf_{L^t,\ldots,L^1;\theta}(p), \quad f_{L^t,\ldots,L^1;\theta} := \text{Re}(L^t \cdots L^3 \theta([L^2, L^1])), $$

where we call $f_{L^t,\ldots,L^1;\theta}$ the $\theta$-dual function of $(L^t, \ldots, L^1)$.

2. For any tower

$$H^{10} = E_0 \supset \ldots \supset E_m, \quad 0 \leq m \leq n, \quad E_k = E_{k-1} \cap \{\omega_k = 0\}, $$

and any choice of vector fields $(L_k^s)$ with $\omega_k = \omega_{L_k^{t_k-1},\ldots,L_k^1;\theta}$, collect all $\theta$-dual functions for all $k$ with $t_k \geq 2$ into the set

$$\{f_{L_k^{t_k-1},\ldots,L_k^1;\theta} : t_k \geq 2\}$$

that we call an associated set of functions of the given tower.
Proposition (used to obtain convexity properties of stratifications)

Let $S \subset \mathbb{C}^{n+1}$ be a smooth real hypersurface and

$$H^{10}S = E_0 \supset \ldots \supset E_m,$$

a tower of the multi-order $(t_1, \ldots, t_n)$ on $S$ with an associated set of functions $\{f_1, \ldots, f_l\}$ ($l \leq m$). Then the following hold:

1. The restrictions to $H^{10}S$ of the differentials $df_1, \ldots, df_l$ are linearly independent, in particular, the zero set

$$M := \{f_1 = \ldots = f_l = 0\} \subset S$$

is a smooth CR submanifold.

2. The kernel (nullspace) distribution (of varying rank) $K^{10} \subset H^{10}S$ of the Levi form of $S$ satisfies

$$H^{10}M \cap K^{10} \subset E_m.$$
Proof of the first structure property

1. It follows from the definition of a tower that the forms \( \omega_1, \ldots, \omega_m \) defined there are linearly independent when restricted to \( H^{10}S \). If the set \( \{ f_1, \ldots, f_l \} \) is empty, (1) is void. Otherwise, the equality of the sets of the forms

\[
\{ \omega_k : t_k \geq 2 \} = \{ df_j |_{H^{10}S} : 1 \leq j \leq l \}
\]

proves (1).

2. To show (2), let \( \xi \in K^{10} \). Since \( \omega_k = \omega_{L_k}; \theta = \theta([\cdot, L_k]) \) when \( t_k = 1 \), it follows that \( \omega_k(\xi) = 0 \). On the other hand, when \( t_k \geq 2 \), \( \xi \in H^{10}M \) implies \( \omega_k(\xi) = df_{l_k}(\xi) = 0 \) for some \( l_k \in \{1, \ldots, l\} \). Hence

\[
\xi \in K^{10} \cap H^{10}M \implies \xi \in H^{10}S \cap \{ \omega_1 = \ldots = \omega_m = 0 \} = E_m
\]

as desired.
Tower multitype: definition

The strata of $S$ are obtained as level sets of the multitype function:

**Definition**

The *tower multitype* of $S$ at $p \in S$ is the CR-invariant

$$T(p) \in (\mathbb{N}_{\geq 2} \cup \{\infty\})^n$$

defined as the *lexicographically minimum multi-order* $(t_1, \ldots, t_n)$ of a tower on a neighborhood of $p$ in $S$.

Here the *lexicographic order* is defined in the standard way by

$$(t_1, \ldots, t_n) < (t'_1, \ldots, t'_n) \iff \exists \ k \in \{1, \ldots, n-1\}, (t_1, \ldots, t_{k-1}) = (t'_1, \ldots, t'_{k-1}), \ t_k < t'_k.$$

Taking the lexicographic order in (3) guarantees that $T(p)$ is an invariant only depending on the CR structure of $S$ (in fact, only on the Levi form).
Recall: \( \theta \)-dual forms \( \omega_{Lt,...,L^1};\theta \) are defined by

\[
\begin{align*}
\omega_{L^1;\theta}(L_p) &:= \theta([L, L^1])(p) & t = 1 \\
\omega_{Lt,...,L^1;\theta}(L_p) &:= L\text{Re}(L_t \cdots L^3\theta([L^2, L^1]))(p), & t \geq 2.
\end{align*}
\]

**Independence of \( \theta \)**

Any other complex contact form satisfies \( \tilde{\theta} = h\theta \), where \( h \) is a nonzero smooth complex function. Then

\[
\omega_{L_{tk}^{-1},...,L_k^1;h\theta} = \omega_{L_{tk}^{-1},...,hL_k^1;\theta}
\]

and with \( \tilde{\theta} \) instead of \( \theta \), we can modify the vector fields \( (L_k^s) \) to obtain the same forms \( \omega_k \) defining the same tower.
Second structure property

Level sets of $\mathcal{T}$ serve as generalized strata, whose properties follow from

**Proposition (used for local inclusion of strata into submanifolds)**

Let $S \subset \mathbb{C}^{n+1}$ be a smooth real hypersurface, $p \in S$ a point, $U \subset S$ an open neighborhood of $p$, and

$$H^{10}U = E_0 \supset \ldots \supset E_m$$

a tower on $U$, whose multi-order equals the multitype $\mathcal{T}(p)$. Choose any associated set of functions

$$\{f_1, \ldots, f_l\}.$$

Then the following hold:

1. $\mathcal{T}(p') \leq \mathcal{T}(p)$ for any $p' \in U$ (with respect to lexicographic order);
2. the tower multitype level set satisfies

$$\{p' \in U : \mathcal{T}(p') = \mathcal{T}(p)\} \subset \{f_1 = \ldots = f_l = 0\}.$$
Proof of the second structure property

Since $\mathcal{T}(p) = (t_1, \ldots, t_n)$ is the multi-order of the given tower on $U$ and $\mathcal{T}(p')$ is the minimum multi-order for a tower on a neighborhood of $p'$, (1) is immediate.

To show (2), choose $p' \in U$ with $f_j(p') \neq 0$ for some $j = 1, \ldots, l$, where

$$f_j(p') = \text{Re}(L_k^{t_{k-1}} \cdots L_k^3 \theta([L_k^2, L_k^1]))(p') \neq 0, \quad L_k^{t_{k-1}}, \ldots, L_k^1 \in E_{k-1} \cup \overline{E}_{k-1}$$

for some $k$ that we choose to be minimal with this property for any $j$. If $L_k^{t_{k-1}} \in \overline{E}_{k-1}$, taking conjugates of all vector fields and of $\theta$ and replacing $\theta$ with $f\theta$, where $f$ is a nonzero function, we may assume that $L_k^{t_{k-1}} \in E_{k-1}$. In particular, we obtain

$$\omega'_k|_{(E_{k-1})p'} \neq 0,$$

where for $x \in S$, $L \in \Gamma(H^{10}S)$,

$$\omega'_k(L_x) := \theta([L, L_k^1])(x) \text{ for } t_k = 3,$$

or

$$\omega'_k(L_x) := LL_k^{t_{k-2}} \cdots L_k^3 \theta([L_k^2, L_k^1]))(x) \text{ for } t_k > 3.$$
Proof of the second structure property, part (2), continued

In the case $t_k = 3$, we have $\omega'_k = \omega_{L^1_k; \theta}$. In the case $t_k > 3$, splitting into real and imaginary parts, we obtain

$$\omega'_k(L_{p'}) = L(\text{Re}f + i\text{Im}f)(p'), \quad f := L^{t_k - 2} \cdots L^3_k \theta([L^2_k, L^1_k])).$$

Taking the term that does not identically vanish for $L_{p'} \in (E_{k-1})_{p'}$ and multiplying $L^{t_k - 2}_k$ by $i$ if necessary, we may assume $\omega'_k(L_x) = L\text{Re}f(x)$, hence $\omega'_k = \omega_{L^{t_k - 2}_k, \ldots, L^1_k; \theta}$. In both cases, we obtain a new tower

$$H^{10}U' = E_0 \supset \ldots \supset E_{k-1} \supset E'_k$$

in a neighborhood $U' \subset U$ of $p'$ of the lexicographically smaller multi-order

$$(t_1, \ldots, t_{k-1}, t_k - 1, \infty, \ldots, \infty) < (t_1, \ldots, t_{k-1}, t_k, \ldots, t_n),$$

by setting

$$E'_k := E_{k-1} \cap \{\omega'_k = 0\}.$$

By definition of the tower multitype, $T(p')$ is the minimum multi-order for a tower in its neighborhood, hence $T(p') < T(p)$ and thus $p'$ is not in the level set of $p$, completing the proof of (2).
Consequences of the second structure property

Since the tower multitype only takes discrete values
\[ T(p) \in (\mathbb{N}_{\geq 2} \cup \{\infty\})^n, \quad p \in S, \]
part (1) of the second structure property immediately yields:

**Corollary**

*For a smooth real hypersurface* \( S \subset \mathbb{C}^{n+1} \), *the following hold:*

1. *The tower multitype function* \( T \) *is upper-semicontinuous.*
2. *Level sets of* \( T \) *are locally closed, i.e. closed in their open neighborhoods.*

**Recall:**

Our goal is to obtain a generalized stratification with convexity properties using level sets of \( T \) as strata:

\[ S = \bigcup_{(t_1, \ldots, t_n) \in (\mathbb{N}_{\geq 2} \cup \{\infty\})^n} \{ p : T(p) = (t_1, \ldots, t_n) \}. \]
Hypersurfaces of finite tower multitype

For simplicity, we shall consider the case \( q = 1 \).

**Theorem (generalized stratification for finite tower multitype)**

Let \( S \subset \mathbb{C}^{n+1} \) be a (not necessarily pseudoconvex) smooth hypersurface whose tower multitype has all entries finite at every point. Then \( S \) is countably 1-regular, where the “strata” can be chosen to be the level sets of the tower multitype function \( T \).

**Recall:**

A hypersurface \( S \subset \mathbb{C}^{n+1} \) is countably 1-regular if it is a countable disjoint union \( S = \bigcup_{k=1}^{\infty} S_k \) of locally closed subsets \( S_k \subset S \) (“strata”) such that for each \( k \) and \( p \in S_k \), there exists a CR submanifold \( M \subset S \) satisfying the following properties:

1. \( M \) contains an open neighborhood of \( p \) in \( S_k \) (in relative topology);
2. \( H_x^{10} M \cap K_x^{10} = \{0\} \) for all \( x \in M \), where \( K_x^{10} \subset H_x^{10} S \) is the kernel of the Levi form of \( S \).
Proof of countable 1-regularity for finite tower multitype

Since the tower multitype of $S$ is finite at every point, $S$ splits into the countable disjoint union of the $T$-level sets

$$S = \bigcup_{(t_1,\ldots,t_n) \in \mathbb{N}^n_{\geq 2}} \{ p : T(p) = (t_1,\ldots,t_n) \}.$$

By the above corollary, each level set of $T$ is locally closed, and by the second structure property, it is locally contained in the zero set

$$M = \{ f_1 = \ldots = f_l = 0 \},$$

where $\{ f_1,\ldots,f_l \}$ is an associated set of functions of a tower

$$H^{10}S = E_0 \supset \ldots \supset E_m$$
on an open subset of $S$.

Since all entries of $T$ are finite, $E_m$ is the zero subbundle. Then by the first structure property, $M$ is a CR submanifold of $S$ satisfying

$$H^{10}M \cap K^{10} \subset E_m = 0,$$

which is precisely the desired convexity property.
Hypersurfaces with subbundles of finite Levi type

We have shown the implication

finite tower multitype $\implies$ countable 1-regularity,

from which known results imply compactness and global regularity. A simpler assumption going back to Kohn’s 1972 JDG paper, is based on

Definition

The Levi type $c(E, p) \in \mathbb{N}_{\geq 2} \cup \{\infty\}$ at $p \in S$ of a subbundle $E \subset H^{10}S$ is

$\min\{t \geq 2 : \exists L^t, \ldots, L^1 \in \Gamma(E) \cup \Gamma(\overline{E}), L^m \cdots L^3 \partial r([L^2, L^1])(p) \neq 0\},$

where $r$ is a local defining function of $S$.

Corollary

For a (not necessarily pseudoconvex) smooth hypersurface $S \subset \mathbb{C}^{n+1}$, assume $c(E, p) < \infty$ for any smooth subbundle $E \subset H^{10}S$ of rank 1 and any $p \in S$. Then $S$ is countably 1-regular.
Proof of the corollary

It suffices to reduce to the case of \textit{finite tower multitype} treated above. Assume by contradiction that for some \( p \in S \), \( T(p) \) is not finite, i.e. \textit{some of the entries are infinite}. By definition, the tower multitype at \( p \) is realized as the multi-order of a tower in a neighborhood \( U \) of \( p \)

\[ H^{10} U = E_0 \supset \ldots \supset E_m. \]

Since not all entries finite, \( E_m \neq \{0\} \).
We claim that \( c(E, p) = \infty \), which will contradict our assumption, hence completing the proof. Indeed, otherwise

\[ L^t \cdots L^3 \theta([L^2, L^1])(p) \neq 0 \]

for some \( t \geq 2 \) and some choice of vector fields \( L^t, \ldots, L^1 \in E_m \cup \overline{E}_m \).

Then, by repeating the arguments of the proof of the second structure property, we reach a contradiction with the tower multitype definition constructing another tower on a neighborhood of \( p \) in \( S \) of a lexicographically smaller multi-order

\[ (t_1, \ldots, t_m, t_{m+1}, \ldots, t_n) < (t_1, \ldots, t_m, \infty, \ldots, \infty). \]
Special subbundles

Our argument yields in fact a stronger more refined version of the above corollary, where the *Levi type* finiteness $c(E, p) < \infty$ only needs to be checked for certain *special subbundles* $E$ that always arise in a tower:

**Definition (special subbundle)**

A complex subbundle $E \subset H^{10}S$ is called *special* if it can be defined by

$$E = \{ \xi \in H^{10}S : \omega_1(\xi) = \ldots = \omega_l(\xi) = 0 \}, \quad \omega_1 \wedge \cdots \wedge \omega_l \neq 0 \text{ on } (H^{10}S)^l,$$

where each $\omega_j, j = 1, \ldots, l$, is the $\theta$-*dual* 1-form $\omega_j = \omega_{L_j^{t_j}} \ldots L_1^1$ for some $t_j \geq 1$ and vector fields $L_j^{t_j}, \ldots, L_1^1 \in \Gamma(H^{10}S) \cup \Gamma(\overline{H^{10}S})$.

**Theorem (finite Levi type only for special subbundles)**

Assume $c(E, p) < \infty$ for any special subbundle $E \subset H^{10}S$ of rank $\geq 1$. Then $S$ is of finite tower multitype at $p$.

Note that $c(E, p) < \infty$ implies $c(E', p) < \infty$ for any subbundle $E' \subset E$. 