February 1, 2020 Lecturer Dmitri Zaitsev Hilary Term 2020

MAU34206 - Harmonic Analysis I 2020

Sheet 1

Due: after the lecture next Wednesday

Exercise 1

Compute Fourier coefficients $\widehat{f}(n)$ for the function $f(x)$, $0 \le x \le 1$, where:

(i) $f(x) = \mathsf{sin}(2\pi x) - 2\mathsf{cos}(10\pi x)$;

(ii) $f(x) = \mathsf{cos}(\pi x)$.

Exercise 2

For $a > 0$:

(i) determine all characters on $G := \mathbb{R}/a\mathbb{Z}$;

(ii) construct an inner product $\langle , \rangle$ on $C([0, a])$ for which the characters form an orthonormal system;

(iii) use the above inner product to compute the Fourier coefficients $\widehat{f}(\chi) = \langle f, \chi \rangle$.

Exercise 3

Let G be an abelian topological group and for $g \in G$ define the translation operator

$$T_g f(x) := f(gx)$$

acting on functions on G.

(i) Find all functions $\chi : G \to \mathbb{T}$ invariant under all T_g, $g \in G$;

(ii) Show that each character $\chi : G \to \mathbb{T}$ is an eigenvector of T_g;

(iii) Let $f : G \to \mathbb{T}$ be an eigenvector of each T_g with eigenvalue $\chi(g)$. Show that $\chi(gh) = \chi(g)\chi(h)$.

(iv) If $\chi(g)$ is as above, show that $f(x) = c\tau(x)$, where c is a constant and $\tau : G \to \mathbb{C}^*$ a homomorphism.

Exercise 4

Recall that discrete topology on a set X is given by the collection of all subsets. Show:

(i) Any set with discrete topology is metrizable.

(ii) Any group with discrete topology is automatically a topological group.

(iii) Construct a non-discrete topology on $\mathbb{Z}$ making it to a topological group. (Hint. Consider homomorphisms into $\mathbb{T}$.)