December 4, 2003 Lecturer Dmitri Zaitsev Michaelmas Term 2003

Course 425 2003-04

S h e e t 5

Due: beginning of the next term

Exercise 1

Give a manifold structure on the n-torus $\mathbb{R}^n/\mathbb{Z}^n$.

Exercise 2

Show that the map

$$t \in \mathbb{R} \mapsto \Big(\cos(\sqrt{2}t)(2+\cos t), \sin(\sqrt{2}t)(2+\cos t), \sin t\Big) \in \mathbb{R}^3$$

defines an immersion whose image S is not a submanifold of $\mathbb{R}^3$. Show that S is dense in the torus

$$\left\{(\cos t(2+\cos s), \sin t(2+\cos s), \sin s) \in \mathbb{R}^3 : s, t \in \mathbb{R}\right\}.$$

Exercise 3

The Veronese and Segre maps are given respectively by

$$v\colon (x,y) \in \mathbb{R}^2 \mapsto (x^n, x^{n-1}y, x^{n-2}y^2, \ldots, x^{n-i}y^i, \ldots, y^n) \in \mathbb{R}^{n+1}$$

and

$$s\colon (x_1, \ldots, x_n, y_1, \ldots, y_m) \in \mathbb{R}^{m+n} \mapsto (x_i y_j)_{1\le i\le n, 1\le j\le m} \in \mathbb{R}^{mn}.$$

Is the image of v (resp. s) a submanifold? If not, what is the maximal open set U in the target space such that the intersection of U with image of v (resp. the image of s) is a submanifold?

Exercise 4

A quadric Q (or hyperquadric) in $\mathbb{R}^n$ is the subset given by $\sum_{ij} a_{ij}x^i x^j + \sum_k b_k x^k + c = 0$, where the symmetric matrix (a_{ij}) is invertible. Show that Q is a submanifold everywhere except possibly one point. What is its dimension?

Exercise 5

Let X be a C^k n-manifold and $x \in X$ a point and let $f_1, \ldots, f_m \in C^k(U)$ be functions defined in an open neighborhood U of x such that their differentials at x are linearly independent. Show that there exists a chart at x whose first m components are the f_i.