Course 414 2007-08

Sheet 4

Due: after the lecture first week of the next term

Exercise 1

Determine the type of singularity (removable, pole, essential or not isolated) and calculate the residue in case it is isolated:

(i) $f(z) = \frac{\sin z}{z-\pi}$ at $z_0 = \pi$;

(ii) $f(z) = \frac{z}{\cos z - 1}$ at $z_0 = 0$;

(iii) $f(z) = ze^{-1/z^3}$ at $z_0 = 0$;

(iv) $f(z) = \frac{z^2}{e^{1/z}-1}$ at $z_0 = 0$;

Exercise 2

For each function f from the previous exercise, determine a maximal open set $\Omega \subset \mathbb{C}$ such that f is meromorphic in Ω.

Exercise 3

Let f have a pole of order m at a point z_0 and g have pole of order n at the same point.

(i) Does $f + g$ always have an isolated singularity at z_0?

(ii) Does $f + g$ always have a pole at z_0?

(iii) Same question for $h = fg$?

(iv) In cases $f + g$ or fg have a pole at z_0, what are the possible pole orders?

Exercise 4

If f and g are entire functions (holomorphic in $\mathbb{C}$) and $|f(z)| \leq |g(z)|$ for all z, show that $f(z) = cg(z)$ for some constant c. (Hint. Consider the function f/g.)

Exercise 5

Suppose that f is meromorphic in $\mathbb{C}$ and bounded outside a disk $B_R(0)$. Show that f is rational. (Hint. Try to eliminate the poles inside the disk.)

Exercise 6

Use Rouché's theorem to find the number of zeroes of the polynomial inside the circle $|z| = 1$:

(i) $z^6 - 5z^4 + z^3 - 2z$;

(ii) $2z^4 - 2z^3 + z^2 - z + 7$;

Exercise 7

Find all Möbius transformations

(i) sending 1, -1, 0 onto 0, 1, ∞ respectively;

(ii) sending 1 and 0 onto 0 and 1 respectively;

(iii) sending 0 to ∞.

Exercise 8

Let $\Omega \subset \overline{\mathbb{C}}$ be open and connected and $f\colon \Omega \to \overline{\mathbb{C}}$ a holomorphic map. Show that either $f \equiv \infty$ or the set $f^{-1}(\infty)$ has no limit points in Ω.

Exercise 9

Find all biholomorphic self-maps of the strip $|\mathsf{Im} z| < 1$. (Hint. Find a biholomorphic map of the strip onto the upper-half plane using e^z.)

Exercise 10

Show that any 3 disjoint points in $\overline{\mathbb{C}}$ belong to a uniquely determined generalized circle (circle or line) in $\overline{\mathbb{C}}$. Use this fact to show the existence of Möbius transformations sending any given generalized circle into any other.