Course 414 2007-08

Sheet 3

Due: after the lecture first week of the next term

Exercise 1

Let γ be the sum of two line segments connecting -1 with iy and iy with 1, where y is a fixed parameter.

(i) Write an explicit parametrization for γ;

(ii) For every y, evaluate the integrals $\int_\gamma z\,dz$ and $\int_\gamma \bar{z}\,dz$.

Which of the integrals is independent of y?

Exercise 2

Let $\Omega := \{z \in \mathbb{C} : 1 < |z| < 5\}$ and set $\gamma_r(t) := re^{it}$, $\lambda(t) := -3 + e^{it}$, $0 \le t \le 2\pi$.

(i) Show that $[\gamma_2] + [\gamma_3]$, $2[\gamma_4]$ and $[\lambda]$ represent cycles in Ω.

(ii) Show that $[\gamma_2] + [\gamma_3]$ and $2[\gamma_4]$ are homologous in Ω.

(iii) Which two of the curves γ_2, γ_3 and λ are homotopic in Ω? Which two induce homologous cycles in Ω? Do the answers change, if Ω is replaced by $\mathbb{C}$?

Hint. Use Cauchy's Theorem to justify that two curves are not homotopic or that two cycles are not homologous.

Exercise 3

Use the theorem on the power series expansion of holomorphic functions to find the radius of convergence of the Taylor series at 0 of the following functions:

(i) $f(z) = \mathsf{Log}(z^2 - iz + 2)$ (the principal value);

(ii) $f(z) = \frac{1}{(z+2)(z-i)\sin(z+\frac{\pi}{2})}$;

Justify your anser.

Exercise 4

Find the maximal open set, where the sequence (f_n) converges compactly (uniformly on every compactum):

(i) $f_n(z) = (z - 1/n)^2$;

(ii) $f_n(z) = z^{n^2}$;

(iii) $f_n(z) = e^{nz}$.

Exercise 5

Let (f_n) and (g_n) be compactly convergent sequences of holomorphic functions in Ω.

(i) Show that the sequences $(f_n + g_n)$ and $(f_n g_n)$ are also compactly convergent in Ω.

(ii) Suppose in addition that g_n has no zeros in Ω for each n. Is the sequence f_n/g_n always compactly convergent in Ω?

Exercise 6

Determine the Laurent series expansion of the function f at a and its ring of convergence:

(i) $f(z) = (z^2 + 1)^{-1}$, $a = i$;

(ii) $f(z) = (z - 1)^{-2} \operatorname{Log} z$, $a = 1$;

(iii) $f(z) = \frac{\cos(\pi z)}{z(z-2)^3}$, $a = 2$;

Exercise 7

Let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ and $g(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n$ be Laurent series converging in a ring $r < |z - z_0| < R$. Find the formula for the Laurent series expansion of the product fg and show that it converges in the same ring.

Exercise 8

Determine the mutliplicity with which f takes its value at z_0:

(i) $f(z) = e^{z \cos z - z}$, $z_0 = 0$;

(ii) $f(z) = (\operatorname{Log}(\cos z))^2$, $z_0 = 2\pi$.

(iii) $f(z) = (1 + z^2 - e^{z^2})^4$, $z_0 = 0$.