December 8, 2005 Lecturer Dmitri Zaitsev Michaelmas Term 2005

**Course 414 2005-06**

Sheet 4

Due: after the lecture at the beginning of the next Term

## Exercise 1

Let $f$ be holomorphic in a convex open set $\Omega$ and satisfy $|f'(z)| \le M$ for all $z \in \Omega$ and some fixed constant $M > 0$.

(i) Prove that $|f(z_1) - f(z_2)| \le M|z_1 - z_2|$ for all $z_1, z_2 \in \Omega$.

(ii) Does the same conclusion hold without the convexity assumption on $\Omega$? Justify your answer by giving a proof or providing a counterexample.

## Exercise 2

Calculate the integral $\int_\gamma f(z)\,dz$ along $\gamma(t) := 2\cos t + i\sin t$, $0 \le t \le 2\pi$, by changing (if necessary) the integration curve to a homotopic one or to a homologous cycle, where calculations are easier, and using the Cauchy's theorem:

(i) $f(z) = \sin(z^2)$;

(ii) $f(z) = 1/z$;

(iii) $f(z) = \frac{1}{z(z-1)}$.

## Exercise 3

Let $\Omega := \{z \in \mathbb{C} : 1 < |z| < 5\}$ and set $\gamma_r(t) := re^{it}$, $\lambda(t) := 2 + e^{it}$, $0 \le t \le 2\pi$.

(i) Show that $[\gamma_2] + [\gamma_3]$, $2[\gamma_4]$ and $[\lambda]$ represent cycles in $\Omega$;

(ii) Show that $[\gamma_2] + [\gamma_3]$ and $2[\gamma_4]$ are homologous in $\Omega$.

(iii) Which two of the curves $\gamma_2$, $\gamma_3$ and $\lambda$ are homotopic in $\Omega$? Which two induce homologous cycles in $\Omega$? Do the answers change, if $\Omega$ is replaced by $\mathbb{C}$?

Hint. Use Cauchy's Theorem to justify that two curves are not homotopic or that two cycles are not homologous.

## Exercise 4

Use the theorem on the power series expansion of holomorphic functions or the Cauchy-Hadamard formula to find the radius of convergence of the Taylor series at 0 of the following functions:

(i) $f(z) = \log(1 + z^2)$;

(ii) $f(z) = \frac{\sin z}{(z-1)(z+2i)}$;