Course 3423 2017

Sheet 2

Due: after the lecture

Exercise 1

Determine the zero order of f at 0:

(i) $f(z) = z\,\mathsf{cos}z - \mathsf{sin}z$;

Solution Use power series expansion

$$z\,\mathsf{cos}z - \mathsf{sin}z = z(1 - \frac{z^2}{2} + O(4)) - (z - \frac{z^3}{3!} + O(5)) = c\neg z^3 + O(4),$$

where $c \neq 0$ and $O(k)$ denotes terms of order greater or equal k. Hence the order of f is 3.

(ii) $f(z) = (\mathsf{Log}(1 + z - \mathsf{sin}\,z))^4$;

Solution Use the expansion

$$\mathsf{Log}(1+w) = w - \frac{w^2}{2} + O(|w|^3)$$

Then

$$(\mathsf{Log}(1 + z - \mathsf{sin}\,z))^4 = ((z - \mathsf{sin}\,z) - \frac{(z - \mathsf{sin}\,z)^2}{2} + O(|z - \mathsf{sin}\,z|^3))^4$$

$$= \left(\frac{z^3}{3!} + O(4)\right)^4 = c\,z^{12} + O(13), \quad c \neq 0,$$

hence the order is 12.

(iii) $f(z) = (1 + z^2 - e^{z^2})^{10}$.

Solution As before,

$$(1 + z^2 - e^{z^2})^{10} = \left(1 + z^2 - (1 + z^2 + \frac{(z^2)^2}{2} + O(|z^2|^3)\right)^{10} = cz^{40} + O(41),$$

hence the order is 40.

Exercise 2

Let (f_n) and (g_n) be compactly convergent sequences of holomorphic functions in Ω.

(i) Show that the sequences (f_n+g_n), $(f_n g_n)$ and $(\mathsf{sin} f_n)$ are also compactly convergent in Ω.

Solution For a compact $K \subset \Omega$, use the estimates

$$\sup |(f_n(z)+g_n(z)) - (f(z)+g(z))| \leq \sup |f_n(z)-f(z)| + \sup |(g_n(z)-g(z)|,$$

$$\sup |(f_n(z)g_n(z)) - (f(z)g(z))| \leq \sup |(f_n(z)-f(z))g_n(z)| + \sup |(f(z)(g_n(z)-g(z))|,$$

where the supremum is taken over K. For $\mathsf{sin} f_n$, observe that the sequence (f_n) has a uniform bound $C > 0$, and use the uniform continuity of $\mathsf{sin} w$ on the disk $|w| \leq C$.

(ii) Is the same conclusion true with "compactly" replaced by "uniformly"?

Solution For uniformly convergent sequences, only the sum is still uniformly convergent.

A counterexample for products is given by

$$\Omega = \mathbb{C}, \quad f_n(z) = g_n(z) := z + \frac{1}{n},$$

where both f_n and g_n converge uniformly to z but their product does not converge uniformly on Ω because

$$\sup_{\mathbb{C}} |f_n g_n| = +\infty.$$

Similarly, for $(\mathsf{sin} f_n)$, the lack of uniform continuity of $\mathsf{sin} w$ for $w \in \mathbb{C}$ can be exploited when the uniform limit of (f_n) is unbounded. Set

$$f_n(z) = iz + i/n.$$

and use the fact that

$$\sup_{x \geq 0} |e^{x+\frac{1}{n}} - e^x| = +\infty$$

to prove that

$$\mathsf{sin}(f_n(z)) - \mathsf{sin}(f(z))$$

is unbounded for $z \in \mathbb{C}$ for every fixed n. The latter implies that $(\mathsf{sin} f_n)$ is not uniformly convergent on $\mathbb{C}$.

(iii) Suppose in addition that g_n has no zeros in Ω for each n. Is the sequence f_n/g_n always compactly convergent in Ω?

Solution No. Take $f_n = z$ and $g_n = 1/n$.

Exercise 3

(i) Show that the sequence $f_n(z) = z^{2n} + z^{n+1}$ converges uniformly on every compact subset of the unit disk $\Omega := \{|z| < 1\}$ but not uniformly on Ω.

Solution For every compact $K \subset \Omega$, one has $c := \sup_K |z| < 1$, hence

$$\sup_K |f_n(z)| \leq c^{2n} + c^{n+1} \to 0, \quad n \to \infty,$$

implying uniform convergence on K to 0.

On the other hand,

$$\sup_\Omega |z^{2n} + z^{n+1}| \geq \sup_{x \geq 0} x^{2n} + x^{n+1} \geq \sup_{x \geq 0} z^{n+1} = 1,$$

hence $f_n(z)$ does not converge uniformly to 0. Further, if there were a function $f(z)$ to which $f_n(z)$ uniformly converges on Ω, then necessarily $f(z) = 0$ by the compact convergence. Therefore (f_n) is not uniformly convergent on Ω (in the sense that there is no function to which it uniformly converges).

(ii) Show the similar property for the power series $\sum_{n=0}^{\infty} (-z)^{n^2+n}$.

Solution Analogous to (i)

Exercise 4

Find the maximal open set, where the sequence (f_n) converges compactly (uniformly on every compactum):

(i) $f_n(z) = (z^2 + \frac{1}{2n})^3$;

Solution The pointwise limit is z^6. Then, for every compact $K \subset \Omega := \mathbb{C}$,

$$|(z^2 + \frac{1}{2n})^3 - z^6| \leq \frac{C}{n}$$

for suitable $C > 0$ depending only on K, which proves the uniform convergence on $\mathbb{C}$. The latter set is clearly maximal.

(ii) $f_n(z) = e^z - \frac{1}{nz}$;

Solution

The pointwise limit is $f(z) := e^z$ for $z \neq 0$. Then for $K \subset \Omega := \mathbb{C} \setminus \{0\}$ compact,

$$|f_n(z) - f(z)| = |(e^z - \frac{1}{nz}) - e^z| \leq \frac{C}{n}$$

proving compact convergence in $\mathbb{C} \setminus \{0\}$. Further, the left-hand side is unbounded on $\mathbb{C}$, hence Ω is maximal.

(iii) $f_n(z) = e^{-n}z$.

Solution $\Omega = \mathbb{C}$, the proof is analogous.

Exercise 5

Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \quad g(z) = \sum_{n=-\infty}^{\infty} b_n(z - z_0)^n$$

be Laurent series converging in a ring $r < |z - z_0| < R$. Find the formula for the Laurent series expansion of the product fg and show that it converges in the same ring.

Solution

The formula for the Laurent series expansion of the product fg is obtained by taking formally the product of both Laurent series:

$$f(z)g(z) = \sum_l \left(\sum_{k+n=l} a_k b_n \right) (z - z_0)^l.$$

To prove the convergence of each

$$c_l := \sum_{k+n=l} a_k b_n,$$

apply the Abel's Lemma separately to both positive and negative parts of each Laurent series, to conclude that for each z in the ring, the series converge absolutely. The last argument also implies that the above series for fg also converges absolutely as desired.

Exercise 6

Give examples of a connected open set (domain) $\Omega \subset \mathbb{C}$ and a holomorphic function f in Ω such that:

(i) Ω is bounded and $f \not\equiv 0$ has infinitely many zeros;

Solution Take

$$\Omega := \{z : \mathsf{Re} z > 0\}, \quad f(z) := \mathsf{sin}\frac{1}{z},$$

then f has zero at each $z = \frac{1}{\pi k}$, $k \in \mathbb{N}$.

(ii) the same as in (i) but f is, in addition, bounded on Ω;

Solution Let Ω and f be as in (i), and take

$$\widetilde{\Omega} := \{z \in \Omega : |f(z)| < 1\}.$$

Then $\widetilde{\Omega}$ satisfies the desired properties.