November, 2017 Lecturer Dmitri Zaitsev Michaelmas Term 2017

Course 3423 2017

Sheet 1

Due: after the lecture

Exercise 1

Give an example of a maximal open set $\Omega \subset \mathbb{C}$, where the given multiple-valued function has a holomorphic branch:

(i) $(1-z)^{1/3}$;

(ii) $\log(z^2+1)$;

(iii) $\sqrt{e^z}$.

Justify your answer.

Solution

(i) $(1-z)^{1/3}$ has the branch

$$f(z) = |1-z|^{1/3} e^{\frac{i\mathrm{Arg}\,(1-z)}{3}}, \quad z \in \Omega := \mathbb{C} \setminus S, \quad S := \{x \in \mathbb{R} : 1 - x \leq 0\}.$$

Here

$$-\pi < \mathrm{Arg}\,(w) \leq \pi$$

is the principal value of the argument, which depends continuously on w away from the ray $\mathbb{R}_{\leq 0}$, but is discontinuous at each point of the ray $\mathbb{R}_{\leq 0}$, and has limits π and $-\pi$ respectively when restricted to the upper and lower half plane. Hence $f(z)$ is indeed continuous on Ω, hence is a branch, and further, the corresponding limits at each $x \in S$ for the restrictions of $f(z)$ differ by the factor of $e^{i\pi/3}$ and hence $f(z)$ cannot be extended to any $x \in S$, proving that Ω is maximal.

Note the other 2 branches given by

$$f_1(z) := e^{i\pi/3} f(z), \quad f_2(z) := e^{2i\pi/3} f(z).$$

(ii) Writing $z^2 + 1 = (z-i)(z+i)$, we see that $\log(z^2+1)$ has infinitely many values

$$f_k(z) := \ln|z^2+1| + i\mathrm{Arg}\,(z^2+1) + 2i\pi k, \quad k \in \mathbb{Z},$$

where

$$z \in \Omega := \mathbb{C} \setminus S, \quad S := \{iy : y \in [-\infty, -1] \cup [1, +\infty]\}.$$

Indeed, $\operatorname{Arg}(w)$ is only discontinuous at $w \in \mathbb{R}_{\leq 0}$, hence $\operatorname{Arg}(z^2+1)$ is only discontinuous at z satisfying $z^2+1 = x \in \mathbb{R}_{\leq 0}$, i.e. at

$$z = \pm\sqrt{x-1} \iff z \in S.$$

Alternatively, we can choose branches individually for the arguments of the factors $z-i$ and $z+i$, defined away from the ray

$$R := \{iy : y \geq -1\},$$

then their sum yields a branch of $\arg((z-i)(z+i))$, which turns out to be continuously extendible to

$$R' := \{iy : y \geq 1\},$$

but not to the interval $[-i, i]$. This will lead to a branch of f defined on the maximal set $\Omega := \mathbb{C} \setminus [-i, i]$.

(iii) The function $\sqrt{e^z}$ has 2 branches, each defined globally on $\mathbb{C}$, given by

$$f_{1,2}(z) = \pm e^{\frac{z}{2}}.$$

Exercise 2

Let $f\colon \Omega \subset \mathbb{C} \to \mathbb{C}$ be continuously $\mathbb{R}$-differentiable with invertible differential at every point. Assume that Ω is connected and f preserves non-oriented angles. Show that either $f_{\bar{z}} = 0$ (i.e. f is holomorphic) or $f_z = 0$ (i.e. f is anti-holomorphic).

Solution Fix a point $a \in \Omega$ and the standard basis $u_1 = 1, u_2 = i$ of $\mathbb{C}$ over $\mathbb{R}$, and set

$$v_j = df_a(u_j), \quad j = 1, 2,$$

where df_a is the differential of f at a.

Then after composing f with a complex-linear map $l(z) = Az$, $A \in \mathbb{C}^*$, we may assume that $v_1 = 1$, $v_2 = \lambda i$ for some $\lambda \in \mathbb{R}^*$. Composing further, if necessary, with the conjugation, we may assume $\lambda > 0$. Then

$$df_a(u_1 + u_2) = 1 + \lambda i,$$

and the angle preservation property implies that $1+\lambda i = t(1+i)$ for some $t \in \mathbb{R}$, whence $\lambda = 1$. This implies that the original differential df_a is ether $\mathbb{C}$-linear or $\mathbb{C}$-antilinear.

Now, we use the continuous dependence of df_a on a to conclude that the set $\Omega' \subset \Omega$ of all a with df_a $\mathbb{C}$-linear is both open and closed in Ω. The closedness is obvious and the openness follows from the assumption that df_a is invertible for each a.

Finally, by connectedness of Ω, it follows that f is either holomorphic or antiholomorphic.

Exercise 3

Show that real and imaginary parts u and v of any holomorphic function f are harmonic, i.e. satisfy the Laplace equation

$$\Delta g = 0, \quad \Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}, \quad g = u + iv.$$

Solution

Direct consequence from the Cauchy-Riemann equations.

Exercise 4

(i) Show that for any complex numbers a, b, the differential operator $D = a\frac{d}{dx} + b\frac{d}{dy}$ is a derivation of the algebra of $\mathbb{R}$-differentiable functions, i.e. show that

$$D(fg) = (Df)g + f(Dg).$$

Solution.

Direct consequence from the Leibnitz rule for the ordinary real partial derivatives.

(ii) Use (i) to show the Leibnitz Rule for the formal derivatives

$$(fg)_z = f_z g + f g_z, \quad (fg)_{\bar{z}} = f_{\bar{z}} g + f g_{\bar{z}}.$$

Solution Follows from (i) since both formal derivatives $\frac{d}{dz}$ and $\frac{d}{d\bar{z}}$ are of the form D as in (i).

(iii) Use the Chain Rule for functions of several real variables to show the same Chain Rule for the formal derivatives:

$$(f \circ g)_z = f_w g_z + f_{\bar{w}} \bar{g}_z, \quad (f \circ g)_{\bar{z}} = f_w g_{\bar{z}} + f_{\bar{w}} \bar{g}_{\bar{z}}$$

Solution Similar to (ii), we can reduce the problem to the partial derivatives in the real variables x and y, i.e. to

$$(f \circ g)_x = f_w g_x + f_{\bar{w}} \bar{g}_x$$

and the similar identity for y. Then writing $g = u + iv$ for the function g and $w = u + iv$ for the variables by a slight abuse of notation, we have

$$(f \circ g)_x = f_u u_x + f_v v_x,$$

and

$$f_w g_x + f_{\bar{w}} \bar{g}_x = \frac{1}{2}(f_u - i f_v)(u_x + i v_x) + \frac{1}{2}(f_u + i f_v)(u_x - i v_x),$$

whose right-hand sides coincide by direct calculation. The proof for y instead of x is similar.

Exercise 5

Determine whether the function f is holomorphic by calculating $f_{\bar{z}}$ using formulas from the previous exercise:

(i) $f(z) = \cos(z^2 + \bar{z}^5)$;

Solution Since $g(w) = \cos(w)$ is holomorphnic,

$$\partial_{\bar{z}} \cos(z^2 + \bar{z}^5) = g_w (z^2 + \bar{z}^5)_{\bar{z}} = g_w (5\bar{z}^4),$$

which is not identically zero, hence f is not holomorphic.

(ii) $f(z) = \overline{e^{\bar{z}^9}}$

Solution We need the derivatives of the conjugation

$$(\bar{z})_z = 0, \quad (\bar{z})_{\bar{z}} = 1.$$

Then begin from inside and compute

$$(\bar{z}^9)_z = 0, \quad (\bar{z}^9)_{\bar{z}} = 9\bar{z}^8.$$

Now since $g(w) = e^w$ is holomorphic, compute

$$(g(\bar{z}^9))_z = 0, \quad (g(\bar{z}^9))_z = 9 g_w \bar{z}^8.$$

Finally, together with the outside conjugation,

$$\partial_{\bar{z}} \overline{g(\bar{z}^9)} = (g(\bar{z}^9))_z = 0,$$

hence f is holomoprhic.

Exercise 6

Let $\Omega := \{z \in \mathbb{C} : 1 < |z| < 5\}$ and set $\gamma_r(t) := re^{it}$, $\lambda(t) := -3 + e^{it}$, $0 \leq t \leq 2\pi$.

(i) Show that $[\gamma_2] - [\gamma_3]$, $2[\gamma_4]$ and $[\lambda]$ represent cycles (chains with zero boundary) in Ω.

(ii) Show that $[\gamma_2] + [\gamma_3]$ and $2[\gamma_4]$ are homologous in Ω.

(iii) Which two of the curves γ_2, γ_3 and λ are homotopic in Ω? Which two induce homologous cycles in Ω? Do the answers change, if Ω is replaced by $\mathbb{C}$?

Hint. Use Cauchy's Theorem to justify that two paths are not homotopic as closed paths or that two cycles are not homologous.

Solution

(i) For every arc $\gamma\colon [0, 2\pi] \to \mathbb{C}$ with $\gamma(0) = \gamma(2\pi)$ and the corresponding 1-chain $[\gamma]$, one has

$$\partial[\gamma] = [\gamma(2\pi)] - [\gamma(0)],$$

hence $[\gamma]$ is a cycle (a chain whose boundary is 0) and sums of cycles are also cycles.

(ii) Here for each $j = 2, 3$, $[\gamma_4] - [\gamma_j]$ bounds an annulus inside Ω that can be triangulated into a sum of 2-chains. Hence $[\gamma_4]?[\gamma_j]$ is the boundary of a 2-chain and hence is null-homologous. Therefore also the sum

$$([\gamma_4] - [\gamma_2]) + ([\gamma_4] - [\gamma_3]) = 2[\gamma_4] - ([\gamma_2] + [\gamma_3])$$

is null-homologous as required.

(iii) We claim that the arcs γ_2 and γ_3 are homotopic to each other as closed arcs but are not homotopic to γ_r in Ω. Further, the cycles $[\gamma_2]$ and $[\gamma_3]$ are homologous to each other but not homologous to $[\lambda]$ in Ω. A homotopy between γ_2 and γ_3 can be obtained via the map

$$H(s, t) := (2 + s)e, \quad 0 \leq t \leq 2\pi, \quad 0 \leq s \leq 1$$

The cycle $[\gamma_2] - [\gamma_3]$ bounds an annulus in Ω that can be triangulated into a sum of 2-chains, hence it is null-homologous, and hence $[\gamma_2]$ is homologous to $[\gamma_3]$.

To show that γ_2 and γ_3 are not homotopic to λ, consider the The integral of $f(z) = 1/z$, which is 0 over γ by the Cauchy?s theorem. On the other hand, the integral of $f(z)$ over γ_2 and γ_3 is $2i\pi$. This proves the claims.

If ? is replaced by C, all arcs become homotopic and all cycles homologous.

Exercise 7

For any path

$$\gamma\colon [a, b] \to \mathbb{C}$$

and any continuous function f on $\gamma([a,b])$, the Cauchy integral is defined by

$$F(z) := \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

(i) Show that F defines a holomorphic function in the complement $\mathbb{C} \setminus \gamma([a,b])$.

Solution

Compute

$$\frac{F(z) - F(z_0)}{z - z_0} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z)(\zeta - z_0)}$$

that converges to

$$\frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)\, d\zeta}{(\zeta - z_0)^2}$$

as $z \to z_0$, hence F is holomorphic.

(ii) Show that $F(z) \to 0$ as $z \to \infty$.

Solution

Use the estimate

$$|F(z)| \leq \frac{\ell(\gamma)}{2\pi} \sup_\Gamma |f(\zeta)| \sup_\Gamma \left| \frac{1}{|\zeta - z|} \right|,$$

where $\Gamma = \gamma([a,b])$ and $\ell(\gamma)$ is the length.

(iii) Give an example of γ and f and a point $z_0 \in \gamma([a,b])$ such that F has no limit at z_0. (Hint. Choose a closed path.)

Solution

Take γ to be the boundary of the unit disk and $f(z) \equiv 1$. Then, for every z fixed, $f(\zeta)/(\zeta - z)$ as function in ζ has the only singularity at $\zeta = z$ with residue $f(z) = 1$. Applying the Residue theorem, we obtain that $F(z) = 1$ whenever z is in the unit disk and $F(z) = 0$ whenever z is outside the unit disk. Hence F has no limit at any point of the boundary of the disk.