Course 3423 2017

Sheet 2

Due: after the lecture Thursday 16 November

Exercise 1

Determine the zero order of f at 0:

(i) $f(z) = z \,\mathsf{cos} z - \mathsf{sin} z$;

(ii) $f(z) = (\mathsf{Log}(1 + z - \mathsf{sin}\, z))^4$;

(iii) $f(z) = (1 + z^2 - e^{z^2})^{10}$.

Exercise 2

Let (f_n) and (g_n) be compactly convergent sequences of holomorphic functions in Ω.

(i) Show that the sequences (f_n+g_n), $(f_n g_n)$ and $(\mathsf{sin} f_n)$ are also compactly convergent in Ω.

(ii) Is the same conclusion true with "compactly" replaced by "uniformly"?

(iii) Suppose in addition that g_n has no zeros in Ω for each n. Is the sequence f_n/g_n always compactly convergent in Ω?

Exercise 3

(i) Show that the sequence $f_n(z) = z^{2n} + z^{n+1}$ converges uniformly on every compact subset of the unit disk $\Omega := \{|z| < 1\}$ but not uniformly on Ω.

(ii) Show the similar property for the power series $\sum_{n=0}^{\infty} (-z)^{n^2+n}$.

Exercise 4

Find the maximal open set, where the sequence (f_n) converges compactly (uniformly on every compactum):

(i) $f_n(z) = (z^2 + \frac{1}{2n})^3$;

(ii) $f_n(z) = e^z - \frac{1}{nz}$;

(iii) $f_n(z) = e^{-n} z$.

Exercise 5

Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n, \quad g(z) = \sum_{n=-\infty}^{\infty} b_n(z - z_0)^n$$

be Laurent series converging in a ring $r < |z-z_0| < R$. Find the formula for the Laurent series expansion of the product fg and show that it converges in the same ring.

Exercise 6

Give examples of a connected open set (domain) $\Omega \subset \mathbb{C}$ and a holomorphic function f in Ω such that:

(i) Ω is bounded and $f \not\equiv 0$ has infinitely many zeros;

(ii) the same as in (i) but f is, in addition, bounded on Ω;