Course 3423 2017

Sheet 1

Due: after the lecture next Wednesday

Exercise 1

Give an example of a maximal open set $\Omega \subset \mathbb{C}$, where the given multiple-valued function has a holomorphic branch:

(i) $(1-z)^{1/3}$;

(ii) $\mathsf{log}(z^2+1)$;

(iii) $\sqrt{e^z}$.

Justify your answer.

Exercise 2

Let $f\colon \Omega \subset \mathbb{C} \to \mathbb{C}$ be continuously $\mathbb{R}$-differentiable with invertible differential at every point. Assume that Ω is connected and f preserves non-oriented angles. Show that either $f_{\bar z} = 0$ (i.e. f is holomorphic) or $f_z = 0$ (i.e. f is anti-holomorphic).

Exercise 3

Show that real and imaginary parts u and v of any holomorphic function f are harmonic, i.e. satisfy the Laplace equation

$$\Delta g = 0, \quad \Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2}, \quad g = u + iv.$$

Exercise 4

(i) Show that for any complex numbers a, b, the differential operator $D = a\frac{d}{dx} + b\frac{d}{dy}$ is a derivation of the algebra of $\mathbb{R}$-differentiable functions, i.e. show that

$$D(fg) = (Df)g + f(Dg).$$

(ii) Use (i) to show the Leibnitz Rule for the formal derivatives

$$(fg)_z = f_z g + f g_z, \quad (fg)_{\bar z} = f_{\bar z} g + f g_{\bar z}.$$

(iii) Use the Chain Rule for functions of several real variables to show the same Chain Rule for the formal derivatives:

$$(f \circ g)_z = f_w g_z + f_{\bar w} \bar g_z, \quad (f \circ g)_{\bar z} = f_w g_{\bar z} + f_{\bar w} \bar g_{\bar z}$$

Exercise 5

Determine whether the function f is holomorphic by calculating $f_{\bar{z}}$ using formulas from the previous exercise:

(i) $f(z) = \cos(z^2 + \bar{z}^5)$;

(ii) $f(z) = \overline{e^{(\bar{z}^9)}}$

Exercise 6

Let $\Omega := \{z \in \mathbb{C} : 1 < |z| < 5\}$ and set $\gamma_r(t) := re^{it}$, $\lambda(t) := -3 + e^{it}$, $0 \leq t \leq 2\pi$.

(i) Show that $[\gamma_2] - [\gamma_3]$, $2[\gamma_4]$ and $[\lambda]$ represent cycles (chains with zero boundary) in Ω.

(ii) Show that $[\gamma_2] + [\gamma_3]$ and $2[\gamma_4]$ are homologous in Ω.

(iii) Which two of the curves γ_2, γ_3 and λ are homotopic in Ω? Which two induce homologous cycles in Ω? Do the answers change, if Ω is replaced by $\mathbb{C}$?

Hint. Use Cauchy's Theorem to justify that two paths are not homotopic as closed paths or that two cycles are not homologous.

Exercise 7

For any path

$$\gamma \colon [a, b] \to \mathbb{C}$$

and any continuous function f on $\gamma([a, b])$, the Cauchy integral is defined by

$$F(z) := \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta.$$

(i) Show that F defines a holomorphic function in the complement $\mathbb{C} \setminus \gamma([a, b])$.

(ii) Show that $F(z) \to 0$ as $z \to \infty$.

(iii) Give an example of γ and f and a point $z_0 \in \gamma([a, b])$ such that F has no limit at z_0. (Hint. Choose a closed path.)