Course 3423 2015

Sheet 2

Due: after the lecture Friday 4 December

Exercise 1

Following the line of the proof of Liouville's Theorem, show that if f is holomorphic in $\mathbb{C}$ and satisfies $|f(z)| \leq A|z|^2 + B|z| + C$ for some fixed $A, B, C > 0$, then f is affine linear, i.e. $f(z) = az^2 + bz + c$ for some $a, b \in \mathbb{C}$.

Exercise 2

Following the line of the proof of the Fundamental Theorem of Algebra, show that if f is holomorphic in $\mathbb{C}$ and satisfies $f(z) \to \infty$ as $z \to \infty$, then there exists z with $f(z) = 0$.

Exercise 3

Determine the type of singularity (removable, pole, essential or not isolated):

(i) $f(z) = \frac{\sin z}{z+\pi}$ at $z_0 = -\pi$;

(ii) $f(z) = \frac{\cos z - 1}{z^2}$ at $z_0 = 0$;

(iii) $f(z) = z^3 e^{1/z}$ at $z_0 = 0$;

(iv) $f(z) = \frac{z^4 - z}{e^{1/z} - 1}$ at $z_0 = 0$;

Exercise 4

For each function f from the previous exercise, determine a maximal open set $\Omega \subset \mathbb{C}$ such that f is meromorphic in Ω.

Exercise 5

Let f have a pole of order m at a point z_0 and g have pole of order n at the same point.

(i) Does $f+g$ always have an isolated singularity at z_0?

(ii) Does $f+g$ always have a pole at z_0?

(iii) Same question for $h = fg$?

(iv) In cases $f+g$ or fg have a pole at z_0, what are the possible pole orders?

Exercise 6

Use Rouché's theorem to find the number of zeroes of the function inside the circle $|z| = 1$:

(i) $f(z) = z^{66} - 19z^7 + z^2 - 2z + e^{z^2}$;

(ii) $f(z) = 2z^{48} - 6z^{73} + z^2 - 38z^{15} + 1 + e^z \sin z$;

Exercise 7

(i) Show that the sequence $f_n(z) = z^{2n} + z^{n+1}$ converges uniformly on every compact subset of the unit disk $\Omega := \{|z| < 1\}$ but not uniformly on Ω.

(ii) Show the similar property for the power series $\sum_{n=0}^{\infty} (-z)^{n^2}$.

Exercise 8

Find the maximal open set, where the sequence (f_n) converges compactly (uniformly on every compactum):

(i) $f_n(z) = (z^2 - \frac{1}{2n})^2$;

(ii) $f_n(z) = e^z - \frac{1}{n}$;

(iii) $f_n(z) = e^{-n} z$.

Exercise 9

Let (f_n) and (g_n) be compactly convergent sequences of holomorphic functions in Ω.

(i) Show that the sequences $(f_n + g_n)$ and $(f_n g_n)$ are also compactly convergent in Ω. Is the same conclusion true with "compactly" replaced by "uniformly"?

(ii) Suppose in addition that g_n has no zeros in Ω for each n. Is the sequence f_n/g_n always compactly convergent in Ω?