Exercise 1
Following the line of the proof of Liouville’s Theorem, show that if \( f \) is holomorphic in \( \mathbb{C} \) and satisfies \( |f(z)| \leq A|z|^2 + B|z| + C \) for some fixed \( A, B, C > 0 \), then \( f \) is affine linear, i.e. \( f(z) = az^2 + bz + c \) for some \( a, b \in \mathbb{C} \).

Exercise 2
Following the line of the proof of the Fundamental Theorem of Algebra, show that if \( f \) is holomorphic in \( \mathbb{C} \) and satisfies \( f(z) \to \infty \) as \( z \to \infty \), then there exists \( z \) with \( f(z) = 0 \).

Exercise 3
Determine the type of singularity (removable, pole, essential or not isolated):

(i) \( f(z) = \frac{\sin z}{z + \pi} \) at \( z_0 = -\pi \);
(ii) \( f(z) = \frac{\cos z - 1}{z^2} \) at \( z_0 = 0 \);
(iii) \( f(z) = z^3 e^{1/z} \) at \( z_0 = 0 \);
(iv) \( f(z) = \frac{z^4 - z}{z^7 + z - 1} \) at \( z_0 = 0 \);

Exercise 4
For each function \( f \) from the previous exercise, determine a maximal open set \( \Omega \subset \mathbb{C} \) such that \( f \) is meromorphic in \( \Omega \).
Exercise 5

Let \( f \) have a pole of order \( m \) at a point \( z_0 \) and \( g \) have pole of order \( n \) at the same point.

(i) Does \( f + g \) always have an isolated singularity at \( z_0 \)?
(ii) Does \( f + g \) always have a pole at \( z_0 \)?
(iii) Same question for \( h = fg \)?
(iv) In cases \( f + g \) or \( fg \) have a pole at \( z_0 \), what are the possible pole orders?

Exercise 6

Use Rouché's theorem to find the number of zeroes of the function inside the circle \( |z| = 1 \):

(i) \( f(z) = z^{66} - 19z^7 + z^2 - 2z + e^{z^2} \);
(ii) \( f(z) = 2z^{48} - 6z^{73} + z^2 - 38z^{15} + 1 + e^z \sin z \);

Exercise 7

(i) Show that the sequence \( f_n(z) = z^{2n} + z^{n+1} \) converges uniformly on every compact subset of the unit disk \( \Omega := \{ |z| < 1 \} \) but not uniformly on \( \Omega \).
(ii) Show the similar property for the power series \( \sum_{n=0}^{\infty} (-z)^n \).

Exercise 8

Find the maximal open set, where the sequence \( (f_n) \) converges compactly (uniformly on every compactum):

(i) \( f_n(z) = (z^2 - \frac{1}{2n})^2 \);
(ii) \( f_n(z) = e^z - \frac{1}{n} \);
(iii) \( f_n(z) = e^{-nz} \).

Exercise 9

Let \( (f_n) \) and \( (g_n) \) be compactly convergent sequences of holomorphic functions in \( \Omega \).

(i) Show that the sequences \( (f_n + g_n) \) and \( (f_n g_n) \) are also compactly convergent in \( \Omega \).
   Is the same conclusion true with “compactly” replaced by “uniformly”?
(ii) Suppose in addition that \( g_n \) has no zeros in \( \Omega \) for each \( n \). Is the sequence \( f_n/g_n \) always compactly convergent in \( \Omega \)?