Course 3423 2015

Sheet 1

Due: after the lecture next Wednesday

Exercise 1

Give an example of a maximal open set $\Omega \subset \mathbb{C}$, where the given multiple-valued function has a holomorphic branch:
(i) $z^{1/3}$;
(ii) $\mathsf{log}(z+1)$;
(iii) $\sqrt{e^z}$.
Justify your answer.

Exercise 2

Let $f\colon \Omega \subset \mathbb{C} \to \mathbb{C}$ be $\mathbb{R}$-differentiable with invertible differential at every point. Assume that f preserves non-oriented angles. Show that either $f_{\bar{z}} = 0$ (i.e. f is holomorphic) or $f_z = 0$ (i.e. f is anti-holomorphic).

Exercise 3

Consider the arcs

$$\gamma_1(t) = t, \quad \gamma_2(t) = it, \quad \gamma_3(t) = (1+i)t, \quad 0 \le t \le 1.$$

Assume that f is $\mathbb{R}$-differentiable at 0 with $df|_0$ invertible and f preserves oriented angles between γ_1, γ_2 and between γ_1, γ_3. Show that f satisfies the Cauchy-Riemann equations at 0.

Exercise 4

Prove the differentiation rules for the formal derivatives:
[(i)] The Leibnitz Rule:

$$(fg)_z = f_z g + f g_z, \quad (fg)_{\bar{z}} = f_{\bar{z}} g + f g_{\bar{z}}.$$

[(ii)] The Chain Rule:

$$(f \circ g)_z = f_w g_z + f_{\bar{w}} \bar{g}_z, \quad (f \circ g)_{\bar{z}} = f_w g_{\bar{z}} + f_{\bar{w}} \bar{g}_{\bar{z}}$$

Exercise 5

Determine whether the function f is holomorphic by calculating $f_{\bar{z}}$ using formulas from the previous exercise:

(i) $f(z) = \cos(z^2\bar{z}^5)$;

(ii) $f(z) = \overline{\sin(\bar{z}^9)}$

Exercise 6

Let $\Omega := \{z \in \mathbb{C} : 1 < |z| < 5\}$ and set $\gamma_r(t) := re^{it}$, $\lambda(t) := -3 + e^{it}$, $0 \le t \le 2\pi$.

(i) Show that $[\gamma_2] + [\gamma_3]$, $2[\gamma_4]$ and $[\lambda]$ represent cycles (chains with zero boundary) in Ω.

(ii) Show that $[\gamma_2] + [\gamma_3]$ and $2[\gamma_4]$ are homologous in Ω.

(iii) Which two of the curves γ_2, γ_3 and λ are homotopic in Ω? Which two induce homologous cycles in Ω? Do the answers change, if Ω is replaced by $\mathbb{C}$?

Hint. Use Cauchy's Theorem to justify that two curves are not homotopic or that two cycles are not homologous.

Exercise 7

Determine the radius of convergence of the power series:

(i) $\sum \frac{(-z)^n}{n^{15}}$;

(ii) $\sum (3^{2n} + (-2)^n) z^{2n}$;

(iii) $\sum (n+1)! z^{2n+1}$.