February 21, 2014 Lecturer Dmitri Zaitsev Michaelmas Term 2013

Course 3423/4 2013-14

Sheet 4

Due: after the lecture first week after the study week

Exercise 1

If f and g are entire functions (holomorphic in $\mathbb{C}$) and $|f(z)| \leq |g(z)|^4$ for all z, show that $f(z) = cg(z)^4$ for some constant c.

Exercise 2

Suppose that f is meromorphic in $\mathbb{C}$ and bounded outside a disk $B_R(0)$. Show that f is rational. (Hint. Try to eliminate the poles inside the disk.)

Exercise 3

Find all Möbius transformations

(i) sending $1, -1, 0$ onto $0, 1, \infty$ respectively;

(ii) preserving the real line $\mathbb{R}$.

Exercise 4

Let $\Omega \subset \overline{\mathbb{C}}$ be open and connected and $f\colon \Omega \to \overline{\mathbb{C}}$ a holomorphic map. Show that either $f \equiv \infty$ or the set $f^{-1}(\infty)$ has no limit points in Ω.

Exercise 5

Find all biholomorphic automorphisms Ω:

(i) $\Omega = \mathbb{C}$.

(ii) $\Omega = \mathbb{C} \setminus \{0\}$.

(Hint. Show that any such automorphism extends to an automorphism of the Riemann sphere.)

Exercise 6

Let $f\colon \Omega \to \overline{\mathbb{C}}$ be holomorphic map into Riemann sphere. Assume f is injective. Show that the inverse f^{-1} exists and is holomorphic on $f(\Omega)$.

Exercise 7

Let f be given by

$$f(z) = c\frac{(z-a)^n}{(z-b)^m}, \quad a \neq b.$$

Assume that f is injective. Show that $n, m \leq 1$.