December 3, 2013 Lecturer Dmitri Zaitsev Michaelmas Term 2013

Course 3423/4 2013-14

Sheet 3

Due: after the lecture first week of the next term

Exercise 1

Determine the Laurent series expansion of the function f at a and its ring of convergence:

(i) $f(z) = \frac{z^2}{z-1}$, $a = 1$;

(ii) $f(z) = (z^2+1)^{-1}$, $a = -i$;

(iii) $f(z) = (z-\pi)^{-3}\cos z$, $a = \pi$;

(iv) $f(z) = \frac{\operatorname{Log} z}{(z-i)^3}$, $a = i$;

Exercise 2

Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$ and $g(z) = \sum_{n=-\infty}^{\infty} b_n(z-z_0)^n$ be Laurent series converging in a ring $r < |z-z_0| < R$. Find the formula for the Laurent series expansion of the product fg and show that it converges in the same ring.

Exercise 3

Determine the zero order of f at z_0:

(i) $f(z) = z\cos z - \sin z$, $z_0 = 0$;

(ii) $f(z) = (\operatorname{Log}(1+z-\sin z))^2$, $z_0 = 2\pi$.

(iii) $f(z) = (1+z^3-e^{z^3})^{10}$, $z_0 = 0$.

Exercise 4

Determine the type of singularity (removable, pole, essential or not isolated):

(i) $f(z) = \frac{\sin z}{z-\pi}$ at $z_0 = \pi$;

(ii) $f(z) = \frac{\cos z - 1}{z^2}$ at $z_0 = 0$;

(iii) $f(z) = z^4 e^{1/z}$ at $z_0 = 0$;

(iv) $f(z) = \frac{z^2}{e^{1/z}-1}$ at $z_0 = 0$;

Exercise 5

For each function f from the previous exercise, determine a maximal open set $\Omega \subset \mathbb{C}$ such that f is meromorphic in Ω.

Exercise 6

Let f have a pole of order m at a point z_0 and g have pole of order n at the same point.

(i) Does $f + g$ always have an isolated singularity at z_0?

(ii) Does $f + g$ always have a pole at z_0?

(iii) Same question for $h = fg$?

(iv) In cases $f + g$ or fg have a pole at z_0, what are the possible pole orders?

Exercise 7

Use Rouché's theorem to find the number of zeroes of the function inside the circle $|z| = 1$:

(i) $f(z) = z^{66} - 10z^5 + z^2 - 2z + e^{z^2}$;

(ii) $f(z) = 2z^{48} - 6z^6 + z^2 - 38z^{15} + 1 + e^z \mathsf{sin} z$;