Exercise 1

Determine the Laurent series expansion of the function \( f \) at \( a \) and its ring of convergence:

(i) \( f(z) = \frac{z^2}{z-1}, \quad a = 1; \)
(ii) \( f(z) = (z^2 + 1)^{-1}, \quad a = -i; \)
(iii) \( f(z) = (z - \pi)^{-3} \cos z, \quad a = \pi; \)
(iv) \( f(z) = \frac{\log z}{(z-i)\pi}, \quad a = i; \)

Exercise 2

Let \( f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \) and \( g(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n \) be Laurent series converging in a ring \( r < |z - z_0| < R \). Find the formula for the Laurent series expansion of the product \( fg \) and show that it converges in the same ring.

Exercise 3

Determine the zero order of \( f \) at \( z_0 \):

(i) \( f(z) = z \cos z - \sin z, \quad z_0 = 0; \)
(ii) \( f(z) = (\log(1 + z - \sin z))^2, \quad z_0 = 2\pi. \)
(iii) \( f(z) = (1 + z^3 - e^{z^3})^{10}, \quad z_0 = 0. \)

Exercise 4

Determine the type of singularity (removable, pole, essential or not isolated):

(i) \( f(z) = \frac{\sin z}{z-\pi} \) at \( z_0 = \pi; \)
(ii) \( f(z) = \frac{\cos z - 1}{z} \) at \( z_0 = 0; \)
(iii) \( f(z) = z^4 e^{1/z} \) at \( z_0 = 0; \)
(iv) \( f(z) = \frac{z^2}{e^{1/z} - 1} \) at \( z_0 = 0; \)
Exercise 5
For each function $f$ from the previous exercise, determine a maximal open set $\Omega \subset \mathbb{C}$ such that $f$ is meromorphic in $\Omega$.

Exercise 6
Let $f$ have a pole of order $m$ at a point $z_0$ and $g$ have pole of order $n$ at the same point.
(i) Does $f + g$ always have an isolated singularity at $z_0$?
(ii) Does $f + g$ always have a pole at $z_0$?
(iii) Same question for $h = fg$?
(iv) In cases $f + g$ or $fg$ have a pole at $z_0$, what are the possible pole orders?

Exercise 7
Use Rouché’s theorem to find the number of zeroes of the function inside the circle $|z| = 1$:
(i) $f(z) = z^{66} - 10z^5 + z^2 - 2z + e^{z^2}$;
(ii) $f(z) = 2z^{48} - 6z^6 + z^2 - 38z^{15} + 1 + e^z \sin z$;