Exercise 1

(i) Show that the sequence $f_n(z) = z + z^{2n}$ converges uniformly on every compact subset of the unit disk $\Omega := \{|z| < 1\}$ but not uniformly on $\Omega$.

(ii) Show the similar property for the power series $\sum_{n=0}^{\infty} z^n$.

Exercise 2

Find the maximal open set, where the sequence $(f_n)$ converges compactly (uniformly on every compactum):

(i) $f_n(z) = (z - \frac{1}{n})^2$;

(ii) $f_n(z) = z - \frac{1}{n}$;

(iii) $f_n(z) = e^{nz}$.

Exercise 3

Let $(f_n)$ and $(g_n)$ be compactly convergent sequences of holomorphic functions in $\Omega$.

(i) Show that the sequences $(f_n + g_n)$ and $(f_ng_n)$ are also compactly convergent in $\Omega$.

Is the same conclusion true with “compactly” replaced by “uniformly”?

(ii) Suppose in addition that $g_n$ has no zeros in $\Omega$ for each $n$. Is the sequence $f_n/g_n$ always compactly convergent in $\Omega$?

Exercise 4

Following the line of the proof of Liouville’s Theorem, show that if $f$ is holomorphic in $\mathbb{C}$ and satisfies $|f(z)| \leq A|z| + B$ for some fixed $A, B > 0$, then $f$ is affine linear, i.e. $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

Exercise 5

Following the line of the proof of the Fundamental Theorem of Algebra, show that if $f$ is holomorphic in $\mathbb{C}$ and satisfies $f(z) \to \infty$ as $z \to \infty$, then the exists $z$ with $f(z) = 0$. 