November 2 | Lecturer Dmitri Zaitsev | Hilary Term 2012

Course 3423/4 2011-12

Sheet 2

Due: after the lecture Wednesday 16 November

Exercise 1

(i) Show that the sequence $f_n(z) = z + z^{2n}$ converges uniformly on every compact subset of the unit disk $\Omega := \{|z| < 1\}$ but not uniformly on Ω.

(ii) Show the similar property for the power series $\sum_{n=0}^{\infty} z^n$.

Exercise 2

Find the maximal open set, where the sequence (f_n) converges compactly (uniformly on every compactum):

(i) $f_n(z) = (z - \frac{1}{n})^2$;

(ii) $f_n(z) = z - \frac{1}{n}$;

(iii) $f_n(z) = e^{nz}$.

Exercise 3

Let (f_n) and (g_n) be compactly convergent sequences of holomorphic functions in Ω.

(i) Show that the sequences $(f_n + g_n)$ and $(f_n g_n)$ are also compactly convergent in Ω. Is the same conclusion true with "compactly" replaced by "uniformly"?

(ii) Suppose in addition that g_n has no zeros in Ω for each n. Is the sequence f_n/g_n always compactly convergent in Ω?

Exercise 4

Following the line of the proof of Liouville's Theorem, show that if f is holomorphic in $\mathbb{C}$ and satisfies $|f(z)| \le A|z| + B$ for some fixed $A, B > 0$, then f is affine linear, i.e. $f(z) = az + b$ for some $a, b \in \mathbb{C}$.

Exercise 5

Following the line of the proof of the Fundamental Theorem of Algebra, show that if f is holomorphic in $\mathbb{C}$ and satisfies $f(z) \to \infty$ as $z \to \infty$, then the exists z with $f(z) = 0$.