February 26, 2010 Lecturer Dmitri Zaitsev Michaelmas Term 2009

Course 3423/4 2009-10

S h e e t 4

Due: after the lecture first week of the next term

Exercise 1

Let f have a pole of order m at a point z_0 and g have pole of order n at the same point.

(i) Does $f+g$ always have an isolated singularity at z_0?
(ii) Does $f+g$ always have a pole at z_0?
(iii) Same question for $h = fg$?
(iv) In cases $f+g$ or fg have a pole at z_0, what are the possible pole orders?

Exercise 2

If f and g are entire functions (holomorphic in $\mathbb{C}$) and $|f(z)| \le |g(z)|^2$ for all z, show that $f(z) = cg(z)$ for some constant c. (Hint. Consider the function f/g^2.)

Exercise 3

Suppose that f is meromorphic in $\mathbb{C}$ and bounded outside a disk $B_R(0)$. Show that f is rational. (Hint. Try to eliminate the poles inside the disk.)

Exercise 4

Use Rouché's theorem to find the number of zeroes of the function inside the circle $|z| = 1$:

(i) $f(z) = z^{11} - 6z^3 + z^2 - 2z$;
(ii) $f(z) = 2z^{44} - 2z^3 + z^2 - 7z + 1$;

Exercise 5

Find all Möbius transformations

(i) sending $1, -1, 0$ onto $0, 1, \infty$ respectively;
(ii) sending 1 and 0 onto 0 and 1 respectively;
(iii) sending 0 to ∞.

Exercise 6

Let $\Omega \subset \overline{\mathbb{C}}$ be open and connected and $f\colon \Omega \to \overline{\mathbb{C}}$ a holomorphic map. Show that either $f \equiv \infty$ or the set $f^{-1}(\infty)$ has no limit points in Ω.

Exercise 7

Find all biholomorphic self-maps of Ω:

(i) $\Omega = \{z : |\mathsf{Re} z| < 1\}$ (Hint. Find a biholomorphic map of the strip onto the upper-half plane using e^z.)

(ii) $\Omega = H \setminus \{i\}$, where H is the upper-half plane.