December 10, 2009 Lecturer Dmitri Zaitsev Michaelmas Term 2009

Course 3423/4 2009-10

Sheet 3

Due: after the lecture first week of the next term

Exercise 1

Consider the sequence of holomorphic functions $f_n(z) = z + \frac{1}{n}$.

(i) Is the sequence (f_n) converging uniformly on $\mathbb{C}$?

(ii) Is the sequence of squares (f_n^2) converging uniformly on $\mathbb{C}$?

(iii) Is the sequence of squares (f_n^2) converging compactly on $\mathbb{C}$?

Justisfy your answer.

Exercise 2

Consider the sequence of real functions on the interval $[0, 1]$ given by $f_n(x) = x^n$.

(i) Show that this sequence is not equvicontinuous on $[0, 1]$.

(ii) Does it become equicontinuous when restricted to the interval $[0, r]$ with $0 < r < 1$? Justify your answer.

Exercise 3

Determine the Laurent series expansion of the function f at a and its ring of convergence:

(i) $f(z) = (z^2 - 1)^{-1}$, $a = -1$;

(ii) $f(z) = (z - 1)^{-3} \sin z$, $a = 1$;

(iii) $f(z) = \frac{\operatorname{Log} z}{(z-2)^3}$, $a = 2$;

Exercise 4

Let $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$ and $g(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n$ be Laurent series converging in a ring $r < |z - z_0| < R$. Find the formula for the Laurent series expansion of the product fg and show that it converges in the same ring.

Exercise 5

Determine the mutliplicity with which f takes its value at z_0:

(i) $f(z) = z \cos z - z$, $z_0 = 0$;

(ii) $f(z) = (\operatorname{Log}(1 + \sin z))^2$, $z_0 = 2\pi$.

(iii) $f(z) = (1 + z^2 - e^{z^2})^2$, $z_0 = 0$.

Exercise 6

Determine the type of singularity (removable, pole, essential or not isolated):

(i) $f(z) = \frac{\sin z}{z+\pi}$ at $z_0 = -\pi$;

(ii) $f(z) = \frac{\cos z - 1}{z^2}$ at $z_0 = 0$;

(iii) $f(z) = z^2 e^{1/z}$ at $z_0 = 0$;

(iv) $f(z) = \frac{z}{e^{1/z}-1}$ at $z_0 = 0$;

Exercise 7

For each function f from the previous exercise, determine a maximal open set $\Omega \subset \mathbb{C}$ such that f is meromorphic in Ω.

Exercise 8

Let $\Omega \subset \mathbb{C}$ be an open set and $K_n \subset \Omega$ a sequence of compact subsets with $\cup_n K_n^\circ = \Omega$, where K_n° is the interior of K_n. Consider the associated seminorms $\|f\|_n := \sup\{|f(z)| : z \in K_n\}$.

(i) Show that $d(f, g) := \sum_n \frac{1}{2^n} \frac{\|f-g\|_n}{1+\|f-g\|_n}$ defines a metric on the space of all holomorphic functions on Ω.

(ii) Show that $f_n \to f$ compactly in Ω if and only if $d(f_n, f) \to 0$ as $n \to \infty$.