Course 2E2 2008-09 (SF Engineers & MSISS & MEMS)

Due: at the end of the tutorial

Sheet 1

Exercise 1

(i) $\mathbf{v}+\mathbf{u}=(-1,3,0)$, $-3\mathbf{v}=(6,-3,0)$, $\|\mathbf{u}\|=\|\mathbf{v}\|=\sqrt{5}$, $\mathbf{u}\cdot\mathbf{v}=0$, angle is $\pi/2$, orthogonal;

(ii) $\mathbf{v}+\mathbf{u}=(2,1,-2,2,0)$, $-3\mathbf{v}=(-3,-3,6,-3,0)$, $\|\mathbf{u}\|=\sqrt{2}$, $\|\mathbf{v}\|=\sqrt{7}$, $\mathbf{u}\cdot\mathbf{v}=2$, angle is $\cos^{-1}\frac{2}{\sqrt{14}}$, not orthogonal;

(iii) $\mathbf{v}+\mathbf{u}=(1,2k,3,7-k)$, $-3\mathbf{v}=(0,-3k,-3,-21)$, $\|\mathbf{u}\|=\sqrt{5+2k^2}$, $\|\mathbf{v}\|=\sqrt{50+k^2}$, $\mathbf{u}\cdot\mathbf{v}=k^2-7k+2$, angle is $\cos^{-1}\frac{k^2-7k+2}{\sqrt{(2k^2+5)(k^2+50)}}$, orthogonal for all k with $k^2-7k+2=0$;

(iv) $\mathbf{v}+\mathbf{u}=(a-2c,a+c,-c,b,d)$, $-3\mathbf{v}=(6c,-3a,3c,0,-3d)$,
$\|\mathbf{u}\|=\sqrt{a^2+b^2+c^2}$, $\|\mathbf{v}\|=\sqrt{a^2+5c^2+d^2}$, $\mathbf{u}\cdot\mathbf{v}=-ac$,
angle is $\cos^{-1}\frac{-ac}{\sqrt{(a^2+b^2+c^2)(a^2+5c^2+d^2)}}$, orthogonal if either $a=0$ or $c=0$;

Exercise 2

(i) $\begin{pmatrix}1&0\\1&1\end{pmatrix}$;

(ii) $\begin{pmatrix}1&0&-3\\0&0&-1\\0&-2&0\end{pmatrix}$;

(iii) $\begin{pmatrix}0&0&0&1\\0&0&1&1\\0&1&1&1\\1&1&1&1\end{pmatrix}$;

(iv) $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$;

(v) $\begin{pmatrix}0&-1&0\\1&0&0\\1&-1&0\\0&-5&0\\0&0&4\end{pmatrix}$.

Sheet 2

Exercise 1

(i) $\begin{pmatrix}1\\1\end{pmatrix}$;

(ii) $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$;

(iii) doesn't make sense.

Exercise 2

(i) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$;

(ii) $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$;

(iii) $\begin{pmatrix} \frac{1}{2} & \frac{-\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{-1-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \end{pmatrix}$;

(iv) $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ -1 \end{pmatrix}$.

Sheet 3

Exercise 1

(i) subspace;

(ii) not subspace;

(iii) subspace.

Exercise 2

(i) don't span;

(ii) span;

(iii) don't span.

Sheet 4

Exercise 1

(i) $x = t$, $y = -2t$, $z = t$;

(ii) $2x + y = 0$, $x - z = 0$;

(iii) $x + y + z = 0$.

(The answers are not unique)

Exercise 2

(i), (ii), (iii), (v) are linearly dependent, (iv) is linearly independent.

Sheet 5

Exercise 1

(ii) is a basis, (i), (iii), (iv), (v), (iv) are not bases.

Exercise 2

(i) $k_1 = \frac{7}{5}$, $k_2 = \frac{4}{3}$;

(ii) $k_1 = \frac{5}{2}$, $c_2 = -\frac{1}{2}$, $c_3 = 4$;

(iii) $k_1 = -3$, $k_2 = 0$, $k_3 = -1$, $k_4 = 2$.

Sheet 6

Exercise 1

(i) row space basis $\{(1,3)\}$, column space basis $\{(-1)\}$, null space basis $\{(3,-1)\}$, dimension is 1 for all;

(ii) row space basis $\{(1)\}$, column space basis $\{\begin{pmatrix} -1 \\ -3 \end{pmatrix}\}$, dimension is 1 for both, null space is zero, it has empty basis, its dimension is 0;

(iii) row space basis $\{(1,-2)\}$, column space basis $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$, dimension is 1 for both, null space basis $\{(2,1)\}$, its dimension is 1;

(iv) row space basis $\{(1,3,0),(0,5,1)\}$, column space basis $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \end{pmatrix}\}$, dimension 2 for both, null space basis $\{(3,1,1)\}$, its dimension is 1;

(v) row space basis $\{(1,2),(1,0)\}$, column space basis $\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}\}$, dimension 2 for both, null space is zero, it has empty basis, its dimension is 0.

(The choice of bases is not unique.)

Exercise 2

(i) $\{\mathbf{u}_1\}$;

(ii) $\{\mathbf{u}_1, \mathbf{u}_2\}$;

Sheet 7

Exercise 1

(i) rank 1, nullity 2;

(ii) rank 3, nullity 0.

Exercise 1

(i) length $\sqrt{3}$, distance $\sqrt{5}$, angle $\pi/2$;

(ii) length $\sqrt{6}$, distance $\sqrt{7}$, angle $\pi/2$.

Sheet 8

Exercise 1

(i) orthogonal but not orthonormal;

(ii) orthogonal but not orthonormal;

(iii) orthogonal and orthonormal.

Exercise 2

(i) $c_1 = 1$, $c_2 = -\frac{2}{5}$, $c_3 = -\frac{1}{5}$;

(ii) $c_1 = 1$, $c_2 = \frac{1}{25}$, $c_3 = -\frac{7}{25}$.

Sheet 9

Exercise 1

(i) $\{(-1, 2), \frac{1}{5}(8, 7)\}$;

(ii) $\{(1, 0, 1), \frac{1}{2}(1, 2, 1), \frac{1}{3}(1, -1, 1)\}$;

(iii) $\{(1, 0, -1, 0), \frac{1}{2}(1, 2, 1, 0), \frac{1}{3}(1, -1, 1, 0), (0, 0, 0, 1)\}$.

Exercise 2

(i) $x = 1/5$;

(ii) $x = y = z = 1/4$.

Sheet 10

Exercise 1

(i) $(\lambda + 2)(\lambda + 1)$;

(ii) $\lambda^2 + 18$;

(iii) $\lambda(\lambda - 1)(\lambda + 3)$;

(iv) $\lambda(\lambda^2 - 2\lambda - 3)$.

Exercise 2

(i) eigenvalues -1 and 5, corresponding eigenvectors $(6, -1)$ and $(0, 1)$;

(ii) eigenvalues 1, 3 and 0, corresponding eigenvectors $(1, 0, 0)$, $(0, 1, 1)$ and $(3, 2, -1)$;

(iii) eigenvalues 1, 3, 0 and 0 (i.e. 0 is counted with multiplicity 2), corresponding eigenvectors $(1, 0, 0, 1)$, $(0, 1, 1, 0)$, $(3, 2, -1, 0)$ and $(0, 0, 0, 1)$ (the last are two linearly independent eigenvectors with eigenvalue 0).

(The eigenvectors are not unique)

Exercise 2

(i) $P = \begin{pmatrix} 6 & -1 \\ 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$;

(ii) $P = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(iii) $P = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Sheet 11

Exercise 1

(i) orthogonal;

(ii) orthogonal;

(iii) not orthogonal;

Sheet 12

Exercise 1

(i) orthogonal;

(ii) orthogonal;

(iii) not orthogonal;

Exercise 2

$1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$;

Sheet 13

Exercise 1

(i) $\frac{1}{2} - \frac{3}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin nx$;

(ii) $\frac{3\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n-1}{n^2} \cos nx - \sum_{n=1}^{\infty} \frac{2(-1)^n-1}{n} \sin nx$.

Exercise 2

Using Euler's formulas, we have:

$$A(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x) \cos wx \, dx, \quad B(w) = \frac{1}{\pi}\int_{-\infty}^{+\infty} f(x) \sin wx \, dx.$$

Splitting the integration and substituting f, we obtain:

$$A(w) = \frac{1}{\pi}\int_{-1}^{1} x \cos wx \, dx = 0$$

since $x \cos wx$ is odd and

$$B(w) = \frac{1}{\pi}\int_{-1}^{1} x \sin wx \, dx = \frac{2}{\pi}\left(\frac{\sin w}{w^2} - \frac{\cos w}{w}\right),$$

where we have used integration by parts. Then the Fourier Integral representation gives

$$f(x) = \int_{0}^{+\infty} (A(w) \cos wx + B(w) \sin wx) \, dw = \frac{2}{\pi}\int_{0}^{+\infty} \left(\frac{\sin w}{w^2} - \frac{\cos w}{w}\right) \sin wx \, dw.$$