February 15, 2008 Lecturer Dmitri Zaitsev Hilary Term 2008

Course 2E2 2007-08 (SF Engineers & MSISS & MEMS)

Due: at the end of the tutorial

Sheet 1

Exercise 1

(i) $\mathbf{v}+\mathbf{u}=(-2,2,0)$, $-5\mathbf{v}=(15,-5,0)$, $\|\mathbf{u}\|=\|\mathbf{v}\|=\sqrt{10}$, $\mathbf{u}\cdot\mathbf{v}=0$, angle is $\pi/2$, orthogonal;

(ii) $\mathbf{v}+\mathbf{u}=(2,2,-1,2,0)$, $-5\mathbf{v}=(-5,-10,5,-5,0)$, $\|\mathbf{u}\|=\sqrt{2}$, $\|\mathbf{v}\|=\sqrt{7}$, $\mathbf{u}\cdot\mathbf{v}=2$, angle is $\cos^{-1}\frac{2}{\sqrt{14}}$, not orthogonal;

(iii) $\mathbf{v}+\mathbf{u}=(-1,2k,3,5-k)$, $-5\mathbf{v}=(0,-5k,-5,-25)$, $\|\mathbf{u}\|=\sqrt{5+2k^2}$, $\|\mathbf{v}\|=\sqrt{26+k^2}$, $\mathbf{u}\cdot\mathbf{v}=k^2-5k+2$, angle is $\cos^{-1}\frac{k^2-5k+2}{\sqrt{(2k^2+5)(k^2+26)}}$, orthogonal for all k with $k^2-5k+2=0$;

(iv) $\mathbf{v}+\mathbf{u}=(a+3c,a-c,-c,b,d)$, $-5\mathbf{v}=(-15c,-5a,5c,0,-5d)$, $\|\mathbf{u}\|=\sqrt{a^2+b^2+c^2}$, $\|\mathbf{v}\|=\sqrt{a^2+10c^2+d^2}$, $\mathbf{u}\cdot\mathbf{v}=2ac$, angle is $\cos^{-1}\frac{2ac}{\sqrt{(a^2+b^2+c^2)(a^2+10c^2+d^2)}}$, orthogonal if either $a=0$ or $c=0$;

Sheet 2

Exercise 1

(i) $\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$;

(ii) $\begin{pmatrix}3 & 0 & -1\\ 0 & 0 & -1\\ 0 & 3 & 0\end{pmatrix}$;

(iii) $\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & 1 & 1\\ 0 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{pmatrix}$;

(iv) $\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$;

(v) $\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ -1 & 1 & 0\\ 2 & 0 & 0\\ 0 & 0 & -7\end{pmatrix}$.

Exercise 2

(i) $\begin{pmatrix}-1\\ 1\end{pmatrix}$;

(ii) $\begin{pmatrix} -1 \\ 6 \end{pmatrix}$;

(iii) doesn't make sense.

Sheet 3

Exercise 1

(i) $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$;

(ii) $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$;

(iii) $\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{-\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1-\sqrt{3}}{2} \\ \frac{1+\sqrt{3}}{2} \end{pmatrix}$;

(iv) $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \\ 1 \end{pmatrix}$.

Exercise 2

(i) subspace;

(ii) not subspace;

(iii) subspace.

Sheet 4

Exercise 1

(i) don't span;

(ii) span;

(iii) span.

Exercise 2

(i) $x = 2t$, $y = -t$, $z = t$;

(ii) $x + 2y = 0$, $y + z = 0$;

(iii) $x + y + 2z = 0$.

(The answers are not unique)

Sheet 5

Exercise 1

(i), (ii), (iii), (v) are linearly dependent, (iv) is linearly independent.

Exercise 2

(ii) is a basis, (i), (iii), (iv), (v), (iv) are not bases.

Sheet 6

Exercise 1

(i) $c_1 = 3$, $c_2 = 4$;

(ii) $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$, $c_3 = 2$;

(iii) $c_1 = -5$, $c_2 = 4$, $c_3 = -\frac{1}{6}$, $c_4 = 2$.

Exercise 2

(i) row space basis $\{(1,2)\}$, column space basis $\{(1)\}$, null space basis $\{(2,1)\}$, dimension 1 for all;

(ii) row space basis $\{(1)\}$, column space basis $\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\}$, dimension 1 for both, null space is zero, it has empty basis, its dimension is 0;

(iii) row space basis $\{(1,2),(-1,2)\}$, column space basis $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\}$, dimension 2 for both, null space is zero, it has empty basis, its dimension is 0;

(iv) row space basis $\{(1,2,0),(-1,2,1)\}$, column space basis $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \end{pmatrix}\}$, dimension 2 for both, null space basis $\{(-2,1,-4)\}$, its dimension is 1;

(v) row space basis $\{(1,-2),(2,0)\}$, column space basis $\{\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}\}$, dimension 2 for both, null space is zero, it has empty basis, its dimension is 0.

(The choice of bases is not unique.)

Sheet 7

Exercise 1

(i) rank 1, nullity 2;

(ii) rank 3, nullity 0.

Exercise 2

(i) $\{\mathbf{u}_1\}$;

(ii) $\{\mathbf{u}_1, \mathbf{u}_2\}$;

Sheet 8

Exercise 1

(i) length $\sqrt{3}$, distance $\sqrt{5}$, angle $\pi/2$;

(ii) length $\sqrt{6}$, distance $\sqrt{14}$, angle $\cos^{-1}\frac{-2}{\sqrt{24}}$.

Exercise 2

(i) orthogonal but not orthonormal;

(ii) orthogonal but not orthonormal;

(iii) orthogonal and orthonormal.

Sheet 9

Exercise 1

(i) $c_1 = -1$, $c_2 = -\frac{1}{25}$, $c_3 = \frac{7}{25}$;

(ii) $c_1 = 1$, $c_2 = \frac{7}{25}$, $c_3 = \frac{1}{25}$.

Exercise 2

(i) $\{(1, -2), \frac{1}{5}(8, 4)\}$;

(ii) $\{(1, 0, 1), \frac{1}{2}(1, 2, -1), (-1, 1, 1)\}$;

(iii) $\{(1, 0, 1, 0), \frac{1}{2}(1, -2, -1, 0), \ldots\}$.

Sheet 10

Exercise 1

(i) $x = 1$;

(ii) $x = 3/2$, $y = -4$;

(iii) $x = y = z = 1/2$.

Exercise 2

(i) $(\lambda + 2)(\lambda + 1)$;

(ii) $\lambda^2 + 18$;

(iii) $\lambda(\lambda - 1)(\lambda + 3)$;

(iv) $\lambda(\lambda^2 - 2\lambda - 3)$.

Sheet 11

Exercise 1

(i) eigenvalues -1 and 5, corresponding eigenvectors $(6, -1)$ and $(0, 1)$;

(ii) eigenvalues 1, 3 and 0, corresponding eigenvectors $(1, 0, 0)$, $(0, 1, 1)$ and $(3, 2, -1)$;

(iii) eigenvalues 1, 3, 0 and 0 (i.e. 0 is counted with multiplicity 2), corresponding eigenvectors $(1, 0, 0, 1)$, $(0, 1, 1, 0)$, $(3, 2, -1, 0)$ and $(0, 0, 0, 1)$ (the last are two linearly independent eigenvectors with eigenvalue 0).

(The eigenvectors are not unique)

Exercise 2

(i) $P = \begin{pmatrix} 6 & -1 \\ 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix}$;

(ii) $P = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & -1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(iii) $P = \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Sheet 12

Exercise 1

π, 14π, $2/n$, π.

Exercise 2

(i) orthogonal;

(ii) orthogonal;

(iii) not orthogonal;

Sheet 13

Exercise 1

(i) $-1 - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \sin nx$;

(ii) $1 + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$;

(iii) $\pi + \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n-1}{n^2} \cos nx$.