Exercise 1

Use Taylor’s formula to find linear and quadratic approximation at \((x_0, y_0) = (0, 0)\):

**Solution.** Taylor’s formula for \(f\) at \((x_0, y_0)\) yields the linear and quadratic approximations \(L(x, y)\) and \(Q(x, y) = L(x, y) + R(x, y)\), where

\[
L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),
\]

\[
R(x, y) = \frac{1}{2}f_{xx}(x_0, y_0)(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{1}{2}f_{yy}(x_0, y_0)(y - y_0)^2.
\]

(i) \(f(x, y) = x^2 e^y\);

**Solution.** For the linear approximation, we need to calculate the value and the first order partial derivatives at \((0, 0)\): \(f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0\). Hence the linear approximation is

\[L(x, y) = 0.\]

For the quadratic approximation, we need to calculate the second order partial derivatives at \((0, 0)\): \(f_{xx}(0, 0) = 2, f_{xy}(0, 0) = f_{yy}(0, 0) = 0\), and hence

\[Q(x, y) = x^2.\]

(ii) \(f(x, y) = x \sin y\);

**Solution.** For the linear approximation we have: \(f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0\).

Hence the linear approximation is

\[L(x, y) = 0.\]

For the quadratic approximation, we have: \(f_{xx}(0, 0) = f_{yy}(0, 0) = 0, f_{xy}(0, 0) = 1\) and hence

\[Q(x, y) = xy.\]
(iii) \( f(x, y) = \frac{1}{1+x+y}; \)

**Solution.** For the linear approximation we have: \( f(0, 0) = 1, f_x(0, 0) = f_y(0, 0) = -1 \). Hence the linear approximation is

\[
L(x, y) = 1 - x - y.
\]

For the quadratic approximation, we have: \( f_{xx}(0, 0) = f_{yy}(0, 0) = 2, f_{xy}(0, 0) = 2 \) and hence

\[
Q(x, y) = 1 - x - y + x^2 + 2xy + y^2.
\]

**Exercise 2**

Give error estimates for the linear approximations in Exercise 1 for

\[-0.1 \leq x \leq 0.1, \quad -0.2 \leq y \leq 0.2.\]

**Solution.** The error estimates come from estimating the error term in the Taylor formula (see Chapter 11.10):

\[
|E| \leq \frac{1}{2} M (|x - x_0| + |y - y_0|)^2
\]

where \( M \) is a bound for the next order derivatives \( |f_{xx}|, |f_{xy}| \) and \( |f_{yy}| \). So we need to calculate these derivatives in each case and estimate them in the range specified. We restrict to (i), the solution for (ii) and (iii) is analogous.

(i) \( f(x, y) = x^2 e^y; \)

**Solution.** We have \( f_{xx} = 2e^y, f_{xy} = 2xe^y, f_{yy} = x^2 e^y \). Then for \( x \) and \( y \) in the above range, we can choose \( M = 2e^{0.2} \) and hence \( |E| \leq \frac{1}{2} 2e^{0.2} (0.1 + 0.2)^2 \).

**Exercise 3**

Find parametric equations for the normal line at the given point:

(i) to the curve \( x^2 + y^3 = 2 \) at \((1, 1)\);

**Solution.** The normal line to a curve \( g(x, y) = 0 \) at the point \((x_0, y_0)\), where \( \nabla g(x_0, y_0) \neq 0 \) is the line passing through \((x_0, y_0)\) in the direction of \( \nabla g(x_0, y_0) = (g_x(x_0, y_0), g_y(x_0, y_0)) \), hence its parametric equations are

\[
x = x_0 + tg_x(x_0, y_0), \quad y = y_0 + tg_y(x_0, y_0).
\]
Now, for \( g = x^2 + y^3 - 2 \) and \((x_0, y_0) = (1, 1)\), we calculate \( g_x(x_0, y_0) = 2\), \( g_y(x_0, y_0) = 3\) and hence the equations are
\[ x = 1 + 2t, \quad y = 1 + 3t, \]
where \( t \) is a free parameter.

(ii) to the surface \( x \cos y + z = 0 \) at \((0, 0, 0)\).

**Solution.** Similarly, the normal line to a surface \( g(x, y, z) = 0 \) at the point \((x_0, y_0, z_0)\), where \( \nabla g(x_0, y_0, z_0) \neq 0 \), is the line passing through \((x_0, y_0, z_0)\) in the direction of the vector \( \nabla g(x_0, y_0, z_0) = (g_x(x_0, y_0, z_0), g_y(x_0, y_0, z_0), g_z(x_0, y_0, z_0)) \), hence its parametric equations are
\[ x = x_0 + t g_x(x_0, y_0, z_0), \quad y = y_0 + t g_y(x_0, y_0, z_0), \quad z = z_0 + t g_z(x_0, y_0, z_0). \]

In our case we obtain \( \nabla g(x_0, y_0, z_0) = (1, 0, 1) \) and hence the parametric equations are
\[ x = t, \quad y = 0, \quad z = t. \]

**Exercise 4**

Sketch the region of integration and evaluate the integral:

(i)
\[ \int_{0}^{1} \int_{-1}^{1} x y \, dx \, dy \]

**Solution.** The region is the rectangular \(-1 \leq x \leq 1, 0 \leq y \leq 1\) and we have
\[ \int_{0}^{1} \int_{-1}^{1} x y \, dx \, dy = \int_{0}^{1} \left( \frac{x^2 y}{2} \right)_{x=-1}^{x=1} \, dy = 0. \]

Evaluation for the other cases is analogous. We only outline the regions.

(ii)
\[ \int_{0}^{1} \int_{0}^{y} (x + y) \, dx \, dy \]

**Solution.** The region is the triangular \( 0 \leq x \leq y, 0 \leq y \leq 1 \).

(iii)
\[ \int_{0}^{1} \int_{0}^{x^2} y \, dy \, dx \]

**Solution.** The region is bounded by the \( x \)-axis, the vertical line \( y = 1 \) and the parabola \( y = x^2 \).