December 9, 2004 Lecturer Dmitri Zaitsev Hilary Term 2005

Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

S h e e t 8

Due: in the tutorial sessions first Wednesday/Thursday of the next term

Exercise 1

Use Taylor's formula to find linear and quadratic approximation at $(x_0, y_0) = (0, 0)$:

Solution. Taylor's formula for f at (x_0, y_0) yields the linear and quadratic approximations $L(x, y)$ and $Q(x, y) = L(x, y) + R(x, y)$, where

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

$$R(x, y) = \frac{1}{2} f_{xx}(x_0, y_0)(x - x_0)^2 + f_{xy}(x_0, y_0)(x - x_0)(y - y_0) + \frac{1}{2} f_{yy}(x_0, y_0)(y - y_0)^2.$$

(i) $f(x, y) = x^2 e^y$;

Solution. For the linear approximation, we need to calculate the value and the first order partial derivatives at $(0, 0)$: $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$. Hence the linear approximation is

$$L(x, y) = 0.$$

For the quadratic approximation, we need to calculate the second order partial derivatives at $(0, 0)$: $f_{xx}(0, 0) = 2$, $f_{xy}(0, 0) = f_{yy}(0, 0) = 0$, and hence

$$Q(x, y) = x^2.$$

(ii) $f(x, y) = x \sin y$;

Solution. For the linear approximation we have: $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$. Hence the linear approximation is

$$L(x, y) = 0.$$

For the quadratic approximation, we have: $f_{xx}(0, 0) = f_{yy}(0, 0) = 0$, $f_{xy}(0, 0) = 1$ and hence

$$Q(x, y) = xy.$$

(iii) $f(x,y) = \frac{1}{1+x+y}$;

Solution. For the linear approximation we have: $f(0,0) = 1$, $f_x(0,0) = f_y(0,0) = -1$. Hence the linear approximation is

$$L(x,y) = 1 - x - y.$$

For the quadratic approximation, we have: $f_{xx}(0,0) = f_{yy}(0,0) = 2$, $f_{xy}(0,0) = 2$ and hence

$$Q(x,y) = 1 - x - y + x^2 + 2xy + y^2.$$

Exercise 2

Give error estimates for the linear approximations in Exercise 1 for

$$-0.1 \leq x \leq 0.1, \quad -0.2 \leq y \leq 0.2.$$

Solution. The error estimates come from estimating the error term in the Taylor formula (see Chapter 11.10):

$$|E| \leq \frac{1}{2} M(|x - x_0| + |y - y_0|)^2$$

where M is a bound for the next order derivatives $|f_{xx}|$, $|f_{xy}|$ and $|f_{yy}|$. So we need to calculate these derivatives in each case and estimate them in the range specified. We restrict to (i), the solution for (ii) and (iii) is analogous.

(i) $f(x,y) = x^2e^y$;

Solution. We have $f_{xx} = 2e^y$, $f_{xy} = 2xe^y$, $f_{yy} = x^2e^y$. Then for x and y in the above range, we can choose $M = 2e^{0.2}$ and hence $|E| \leq \frac{1}{2}2e^{0.2}(0.1 + 0.2)^2$.

Exercise 3

Find parametric equations for the normal line at the given point:

(i) to the curve $x^2 + y^3 = 2$ at $(1,1)$;

Solution. The normal line to a curve $g(x,y) = 0$ at the point (x_0, y_0), where $\nabla g(x_0, y_0) \neq 0$ is the line passing through (x_0, y_0) in the direction of $\nabla g(x_0, y_0) = (g_x(x_0, y_0), g_y(x_0, y_0))$, hence its parametric equations are

$$x = x_0 + tg_x(x_0, y_0), \quad y = y_0 + tg_y(x_0, y_0).$$

Now, for $g = x^2 + y^3 - 2$ and $(x_0, y_0) = (1,1)$, we calculate $g_x(x_0, y_0) = 2$, $g_y(x_0, y_0) = 3$ and hence the equations are

$$x = 1 + 2t, \quad y = 1 + 3t,$$

where t is a free parameter.

(ii) to the surface $x \cos y + z = 0$ at $(0, 0, 0)$.

Solution. Similarly, the normal line to a surface $g(x, y, z) = 0$ at the point (x_0, y_0, z_0), where $\nabla g(x_0, y_0, z_0) \neq 0$, is the line passing through (x_0, y_0, z_0) in the direction of the vector $\nabla g(x_0, y_0, z_0) = (g_x(x_0, y_0, z_0), g_y(x_0, y_0, z_0), g_z(x_0, y_0, z_0))$, hence its parametric equations are

$$x = x_0 + t g_x(x_0, y_0, z_0), \quad y = y_0 + t g_y(x_0, y_0, z_0), \quad z = z_0 + t g_z(x_0, y_0, z_0).$$

In our case we obtain $\nabla g(x_0, y_0, z_0) = (1, 0, 1)$ and hence the parametric equations are

$$x = t, \quad y = 0, \quad z = t.$$

Exercise 4

Sketch the region of integration and evaluate the integral:

(i)

$$\int_0^1 \int_{-1}^1 xy \, dx \, dy$$

Solution. The region is the rectangular $-1 \leq x \leq 1$, $0 \leq y \leq 1$ and we have

$$\int_0^1 \int_{-1}^1 xy \, dx \, dy = \int_0^1 \left(\frac{x^2 y}{2} \Big|_{x=-1}^{x=1} \right) dy = 0.$$

Evaluation for the other cases is analogous. We only outline the regions.

(ii)

$$\int_0^1 \int_0^y (x + y) \, dx \, dy$$

Solution. The region is the triangular $0 \leq x \leq y$, $0 \leq y \leq 1$.

(iii)

$$\int_0^1 \int_0^{x^2} y \, dy \, dx$$

Solution. The region is bounded by the x-axis, the vertical line $y = 1$ and the parabola $y = x^2$.