Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

S h e e t 7

Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Use the method of Lagrange multipliers to find the local extreme values (local maxima and local minima) of the function f subject to the constraint:

Solution. We introduce the multiplier λ and solve the system $\nabla f = \lambda \nabla g$, $g = 0$.

(i) $f(x, y) = xy$ on the ellipse $x^2 + 4y^2 = 1$;
Here $\nabla f = (y, x)$, $\nabla g = (2x, 8y)$, the system is

$$y = 2\lambda x, \quad x = 8\lambda y, \quad x^2 + 4y^2 - 1 = 0.$$

Substitution of $2\lambda x$ for y yields $x = 16\lambda^2 x$ or $x(1 - 16\lambda^2) = 0$. The latter has solutions $x = 0$ or $\lambda = \pm 1/4$.

Case 1. $x = 0$. Substitution into the system yields $y = 0$, $-1 = 0$, a contradiction.

Case 2. $\lambda = 1/4$. Substitution into the system yields $y = x/2$, $2x^2 - 1 = 0$, $(x, y) = \pm(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}})$.

Case 3. $\lambda = -1/4$. Analogously $(x, y) = \pm(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}})$.

We have

$$f(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}) = \frac{1}{4}, \quad f(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}) = -\frac{1}{4}.$$

The extreme values are $\pm 1/4$.

(ii) $f(x, y) = x^2 y$ on the line $x + y = 3$;
Here $\nabla f = (2xy, x^2)$, $\nabla g = (1, 1)$, the system is

$$2xy = \lambda, \quad x^2 = \lambda, \quad x + y - 3 = 0.$$

Substitute $2xy$ for λ: $x(2y - x) = 0$, $x + y - 3 = 0$. The solutions are $(0, 3)$ and $(2, 1)$, the corresponding extreme values $f(0, 3) = 0$, $f(2, 1) = 4$.

(iii) $f(x,y) = 2x - y + 6$ on the circle $x^2 + y^2 = 4$;

Here $\nabla f = (2, -1)$, $\nabla g = (2x, 2y)$, the system is

$$2 = 2\lambda x, \quad -1 = 2\lambda y, \quad x^2 + y^2 - 4 = 0.$$

From the first equation $\lambda \neq 0$, hence we can solve $x = 1/\lambda$, $y = -1/2\lambda$. Substituting in the third equation and solving yields $\lambda = \pm\sqrt{5}/4$ and hence $(x,y) = (\frac{4}{\sqrt{5}}, -\frac{2}{\sqrt{5}})$ or $(x,y) = (-\frac{4}{\sqrt{5}}, \frac{2}{\sqrt{5}})$. The extreme values are obtained by evaluating f at these points.

(iv) $f(x,y,z) = xyz$ on the plain $x + y + z = 1$.

Here $\nabla f = (yz, xz, xy)$, $\nabla g = (1, 1, 1)$, the system is

$$yz = \lambda, \quad xz = \lambda, \quad xy = \lambda, \quad x + y + z - 1 = 0.$$

From the first equation $\lambda = yz$, substitute in the second: $xz = yz$ or $(x - y)z = 0$.

Case 1. $z = 0$ yields $\lambda = 0$, $xy = 0$ with two solutions $(0, 1, 0)$ and $(1, 0, 0)$, where the value of f is 0.

Case 2. $z \neq 0$ yields $x = y$ and hence $yz = \lambda = xy$ yields $y(z - y) = 0$. If $y = 0$, it follows from the last equation that $z = 1$ and we have the solution $(0, 0, 1)$ with the value of f being 0. Finally, if $y = z$, we have $x = y = z = 1/3$ and the solution $(1/3, 1/3, 1/3)$ with the value of f being $1/27$.

Hence we have two possible extreme values 0 and $1/27$.

(v) $f(x,y,z) = x + 2y + 3z$ on the sphere $x^2 + y^2 + z^2 = 25$.

Here $\nabla f = (1, 2, 3)$, $\nabla g = (2x, 2y, 2z)$, the system is

$$1 = 2\lambda x, \quad 2 = 2\lambda y, \quad 3 = 2\lambda z, \quad x^2 + y^2 + z^2 - 25 = 0.$$

Exercise 2

Minimize the function f subject to two constraints:

Solution. Here we have two constraints $g_1 = 0$ and $g_2 = 0$ and thus need two multipliers λ and μ and the system becomes $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$, $g_1 = 0$, $g_2 = 0$.

(i) $f(x,y,z) = xyz$ on the intersection of $x^2 + y^2 - 1 = 0$ and $x - z = 0$;

Solution. The system becomes

$$(yz, xz, xy) = \lambda(2x, 2y, 0) + \mu(1, 0, -1), \quad x^2 + y^2 - 1 = 0, \quad x - z = 0$$

or

$$\begin{cases} yz = 2\lambda x + \mu \\ xz = 2\lambda y \\ xy = -\mu \\ x^2 + y^2 - 1 = 0 \\ x - z = 0 \end{cases}$$

Expressing μ from the third and z from the last equation and substuting, we get

$$\begin{cases} z = x \\ \mu = -xy \\ 2xy = 2\lambda x \\ x^2 = 2\lambda y \\ x^2 + y^2 - 1 = 0. \end{cases}$$

Next we want to eliminate λ but we have to divide e.g. by x in the third equation. This is only allowed if $x \neq 0$, so we have to treat cases:

Case 1. $x = 0$ which yields $\lambda = 0$ and $y = \pm 1$, hence two points $(0, \pm 1, 0)$. The value of f at both points is 0.

Case 2. $x \neq 0$ and we get $\lambda = y$, hence $x^2 = 2y^2$ from the 4th equation. Substitution into the last one yields $3y^2 = 1$ or $y = \pm 1/\sqrt{3}$ and $z = x = \pm\sqrt{2/3}$. Here the signs for x and y can be chosen independently and $z = x$ from the first equation. Hence we obtain 4 solutions:

$$(\sqrt{2/3}, 1/\sqrt{3}, \sqrt{2/3}), \quad (\sqrt{2/3}, -1/\sqrt{3}, \sqrt{2/3}),$$

$$(-\sqrt{2/3}, 1/\sqrt{3}, -\sqrt{2/3}), \quad (-\sqrt{2/3}, -1/\sqrt{3}, -\sqrt{2/3})$$

leading to the values of f equal to $\pm\frac{2}{3\sqrt{3}}$. The required minimum is the minimum of the values obtained which is $-\frac{2}{3\sqrt{3}}$.

(ii) $f(x, y, z) = x^2 + y^2 + z^2$ on the intersection of $y + 4z - 4 = 0$ and $4y^2 - z^2 = 0$.
Solution. Now the required system becomes

$$(2x, 2y, 2z) = \lambda(0, 1, 4) + \mu(0, 8y, -2z), \quad y + 4z - 4 = 0, \quad 4y^2 - z^2 = 0$$

or

$$\begin{cases} 2x = 0 \\ 2y = \lambda + 8y\mu \\ 2z = 4\lambda - 2z\mu \\ y + 4z - 4 = 0 \\ 4y^2 - z^2 = 0. \end{cases}$$

The first equation gives $x = 0$ and the last two equations give two solutions for y and z. The values of f at the points obtained are to be compared and the minimal value is the desired minimum.

Note that f has no maximum under the given constraints because x^2 can be arbitrarily large.