November 23, 2004 Lecturer Dmitri Zaitsev Hilary Term 2005

Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

S h e e t 6

Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Find the linearization ($L(x, y)$ or $L(x, y, z)$) of the function at the given point:
(i) $f(x, y) = x^2 + y^2 - 1$ at $(-1, 1)$;
(ii) $f(x, y) = e^x \cos y$ at $(0, \pi)$;
(iii) $f(x, y, z) = x^2 + y^2 + z^2$ at $(1, 1, 1)$;
(iv) $f(x, y, z) = \sqrt{x + y + z}$ at $(1, 0, 0)$.

Solution. Use the formula

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

for two variables (x, y) or

$$L(x, y, z) =$$
$$f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0)$$

for three variables (x, y, z):
(i) $f(x, y) = x^2 + y^2 - 1$ at $(-1, 1)$:
$L(x, y) = 1 - 2(x + 1) + 2(y - 1) = -3 - 2x + 2y$.
(ii) $f(x, y) = e^x \cos y$ at $(0, \pi)$;
$L(x, y) = -1 - 1(x - 0) + 0(y - \pi) = -1 - x$.
(iii) $f(x, y, z) = x^2 + y^2 + z^2$ at $(1, 1, 1)$;
$L(x, y, z) = 3 + 2(x - 1) + 2(y - 1) + 2(z - 1) = -3 + 2x + 2y + 2z$.
(iv) $f(x, y, z) = \sqrt{x + y + z}$ at $(1, 0, 0)$.
$L(x, y, z) = 1 + \frac{1}{2}(x - 1) + \frac{1}{2}y + \frac{1}{2}z$.

Exercise 2

Find all the local maxima, local minima, and saddle points of the functions:
(i) $f(x, y) = x^2 - 2x + y^2 + 2y + 3$;

Solution. The first derivative test $f_x = f_y = 0$ yields:

$$2x - 2 = 0, \quad 2y + 2 = 0 \quad \Rightarrow \quad (x, y) = (1, -1).$$

The second derivative matrix at $(x, y) = (1, -1)$ is

$$\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}\Bigg|_{(1,-1)} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $f_{xx} = 2 > 0$ and $f_{xx}f_{yy} - (f_{xy})^2 = 4 > 0$, we have local minimum at $(1, -1)$.

(ii) $f(x, y) = x^2 + xy - y^2$;

Solution. The first derivative test $f_x = f_y = 0$ yields:

$$2x + y = 0, \quad x - 2y = 0 \quad \Rightarrow \quad (x, y) = (0, 0).$$

At $(x, y) = (0, 0)$ we have $f_{xx}f_{yy} - (f_{xy})^2 = 2 \cdot 2 - 1^2 = -5 < 0$, hence $(0, 0)$ is a saddle point.

(iii) $f(x, y) = x^2 + y^3 - 6y + 3$;

Solution. The first derivative test $f_x = f_y = 0$ yields:

$$2x = 0, \quad 3y^2 - 6y = 0 \quad \Rightarrow \quad (x, y) = (0, \pm\sqrt{2}).$$

At $(x, y) = (0, \sqrt{2})$ we have $f_{xx} = 2 > 0$, $f_{xx}f_{yy} - (f_{xy})^2 = 2 \cdot 6y - 0^2 = 12\sqrt{2} > 0$, hence local minimum. At $(x, y) = (0, -\sqrt{2})$ we have $f_{xx}f_{yy} - (f_{xy})^2 = 2 \cdot 6y - 0^2 = -12\sqrt{2} < 0$, hence saddle point.

(iv) $f(x, y) = x^4 + y^4 + 4xy$;

Solution. The first derivative test $f_x = f_y = 0$ yields:

$$4x^3 + 4y = 0, \quad 4y^3 + 4x = 0 \quad \Rightarrow \quad (x, y) = (0, 0) \text{ or } (x, y) = (\pm 1, \mp 1).$$

At $(x, y) = (0, 0)$ we have $f_{xx}f_{yy} - (f_{xy})^2 = 0 \cdot 0 - 4^2 < 0$, hence saddle point. At $(x, y) = (-1, 1)$ we have $f_{xx} = 12 > 0$, $f_{xx}f_{yy} - (f_{xy})^2 = 12 \cdot 12 - 4^2 > 0$, hence local minimum. At $(x, y) = (1, -1)$ we have again $f_{xx} = 12 > 0$, $f_{xx}f_{yy} - (f_{xy})^2 = 12 \cdot 12 - 4^2 > 0$, hence local minimum.

(v) $f(x,y) = \frac{1}{\sqrt{1-x^2-y^2}}$;

Solution. The first derivative test $f_x = f_y = 0$ yields:

$$\frac{x}{(1-x^2-y^2)^{3/2}} = 0, \quad \frac{y}{(1-x^2-y^2)^{3/2}} = 0 \quad \Rightarrow \quad (x,y) = (0,0).$$

At $(x,y) = (0,0)$ we have $f_{xx} = 1 > 0$, $f_{xx}f_{yy} - (f_{xy})^2 = 1 \cdot 1 - 0^2 > 0$, hence local minimum.

(vi) $f(x,y) = x^2 + \sin y$.

Solution. The first derivative test $f_x = f_y = 0$ yields:

$$2x = 0, \quad \cos y = 0 \quad \Rightarrow \quad (x,y) = (0, \pm\frac{\pi}{2} + 2\pi k), \quad k = 0, \pm 1, \pm 2, \ldots.$$

At $(x,y) = (0, \frac{\pi}{2} + 2\pi k)$ we have $f_{xx}f_{yy} - (f_{xy})^2 = 1 \cdot (-1) - 0^2 < 0$, hence saddle point. At $(x,y) = (0, -\frac{\pi}{2} + 2\pi k)$ we have $f_{xx} = 1 > 0$, $f_{xx}f_{yy} - (f_{xy})^2 = 1 \cdot 1 - 0^2 > 0$, hence local minimum.