Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

S h e e t 5

Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Find the gradient of the function:

(i) $f(x, y) = x - y$;

(ii) $f(x, y) = e^{x-y}$;

(ii) $f(x, y, z) = x(\mathsf{cos}y + \mathsf{sin}z)$;

Solution

(i) $f(x, y) = x - y$: $\nabla f = (f_x, f_y) = (1, -1)$;

(ii) $f(x, y) = e^{x-y}$: $\nabla f = (e^{x-y}, -e^{x-y})$;

(ii) $f(x, y, z) = x(\mathsf{cos}y + \mathsf{sin}z)$: $\nabla f = (f_x, f_y, f_z) = (\mathsf{cos}y + \mathsf{sin}z, -x\mathsf{sin}y, x\mathsf{cos}z)$;

Exercise 2

Find the derivative of the function f at the point P_0 in the direction of the vector $\mathbf{a}$:

(i) $f(x, y) = x - y, \quad P_0(1, 0), \quad \mathbf{a} = (1, 1)$;

(ii) $f(x, y) = x^2 + y^2, \quad P_0(-1, -1), \quad \mathbf{a} = (-1, 2)$;

(iii) $f(x, y, z) = 2e^x \mathsf{cos}(yz), \quad P_0(0, 0, 0), \quad \mathbf{a} = (1, 0, 1)$.

Solution

Use the formula $D_{\mathbf{u}} f = \nabla f \cdot \mathbf{u}$ for the directional derivative with respect to the unit vector $\mathbf{u} = \frac{\mathbf{a}}{|\mathbf{a}|}$:

(i) $\nabla f|_{(1,0)} = (1, -1)$, $D_{\mathbf{u}} f = 0$.

(ii) $\nabla f|_{(-1,-1)} = (2x, 2y)|_{(-1,-1)} = (-2, -2), \qquad D_{\mathbf{u}} f = (-2, -2) \cdot \frac{(-1,2)}{\sqrt{1^2+2^2}} = \frac{-2}{\sqrt{5}}$.

(iii) $\nabla f|_{(0,0,0)} = (2e^x \mathsf{cos}(yz), 2ze^x \mathsf{cos}(yz), 2e^y \mathsf{cos}(yz))|_{(0,0,0)} = (2, 0, 0)$,

$D_{\mathbf{u}} f = (2, 0, 0) \cdot \frac{(1,0,1)}{\sqrt{1^2+1^2}} = \frac{2}{\sqrt{2}}$.

Exercise 3

Find tangent line to the curve given by the equation at the given point P_0:

(i) $x^2 + y^2 = 8, \quad P_0(-2, 2)$;

(ii) $xy = -1, \quad P_0(1, -1)$.

Solution Use the formula

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0$$

for the tangent line to the curve $f(x, y) = c$ at the point (x_0, y_0) on the curve:

(i) $f(x, y) = x^2 + y^2$, $(x_0, y_0) = (-2, 2)$.

We have $(f_x, f_y)|_{(-2,2)} = (2x, 2y)_{(-2,2)} = (-4, 4)$ and the tangent line is $-4(x + 2) + 4(y - 2) = 0$.

(ii) $f(x, y) = xy$, $(x_0, y_0) = (1, -1)$.

We have $(f_x, f_y)|_{(1,-1)} = (y, x)_{(1,-1)} = (-1, 1)$ and the tangent line is $-(x - 1) + (y + 1) = 0$.

Exercise 4

Find tangent plane to the surface given by the equation at the given point P_0:

(i) $x^2 + y^2 + z^3 = 3, \quad P_0(-1, -1, 1)$;

(ii) $z - 2x^2 = 0, \quad P_0(1, 1, -2)$.

(iii) $\cos \pi x - yz = 0, \quad P_0(0, 1, 1)$.

Solution

Use the formula

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

for the tangent plane to the surface $f(x, y, z) = c$ at the point (x_0, y_0, z_0) on the surface:

(i) $f(x, y, z) = x^2 + y^2 + z^3$, $(x_0, y_0, z_0) = (-1, -1, 1)$.

We have $(f_x, f_y, f_z)|_{(-1,-1,1)} = (2x, 2y, 3z^2)|_{(-1,-1,1)} = (-2, -2, 3)$ and the tangent plane is $-2(x + 1) - 2(y + 1) + 3(z - 1) = 0$ or $-2x - 2y + 3z - 7 = 0$.

(ii) $f(x, y, z) = z - 2x^2 = 0$, $(x_0, y_0, z_0) = P_0(1, 1, -2)$.

We have $(f_x, f_y, f_z)|_{(1,1,-2)} = (-4x, 0, 1)|_{(1,1,-2)} = (-4, 0, 1)$ and the tangent plane is $-4(x - 1) + 0(y - 1) + (z + 2) = 0$ or $-4x + z + 6 = 0$.

(iii) $y + z = 2$.