Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

Sheet 4

Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Use Chain Rule to express $\frac{dw}{dt}$ as a function of t in the following cases:

(i) $w = x + y^2$, $x = \cos t$, $y = \sin t$;

(ii) $w = \frac{x}{y}$, $x = e^t$, $y = \sin t$;

(iii) $w = \ln(x - y + z)$, $x = \cos t$, $y = \sin t$, $z = \sqrt{t}$;

(iv) $w = z + \sin(xy)$, $x = t$, $y = \ln t$, $z = t^2$.

Solution

Using the Chain Rule $\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$ we obtain:

(i) $\frac{dw}{dt} = 1 \cdot (-\sin t) + 2y\cos t = -\sin t + 2\sin t\cos t$;

(ii) $\frac{dw}{dt} = \frac{1}{y}e^t - \frac{x}{y^2}\cos t = \frac{e^t}{\sin t} - \frac{e^t\cos t}{\sin^2 t}$.

In the following use the Chain Rule in three variables x, y, z:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt},$$

(iii) $\frac{dw}{dt} = \frac{1}{x-y+z}(-\sin t) - \frac{1}{x-y+z}\cos t + \frac{1}{x-y+z}\frac{1}{2\sqrt{t}} = \frac{1}{\cos t - \sin t + \sqrt{t}}\left(-\sin t - \cos t + \frac{1}{2\sqrt{t}}\right)$;

(iv) $\frac{dw}{dt} = y\cos(xy) \cdot 1 + x\cos(xy)\frac{1}{t} + 1 \cdot 2t = (\ln t)\cos(t\ln t) + \cos(t\ln t) + 2t$.

Exercise 2

Use Chain Rule to express $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ as functions of r and θ in the following cases:

(i) $z = xe^y$, $x = r\cos\theta$, $y = r\sin\theta$;

(ii) $z = \frac{x}{y}$, $x = r\cos\theta$, $y = r\sin\theta$;

(iii) $z = x^2 + y^2 + u^2$, $x = r\cos\theta$, $y = r\sin\theta$, $u = r$.

Solution

Using the Chain Rule $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$, $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta}$:

(i) $\frac{\partial z}{\partial r} = e^y\cos\theta + xe^y\sin\theta = e^{r\sin\theta}\cos\theta + r\cos\theta e^{r\sin\theta}\sin\theta$,

$\frac{\partial z}{\partial \theta} = e^y(-r\sin\theta) + xe^y r\cos\theta = -r\sin\theta e^{r\sin\theta} + r^2\cos^2\theta e^{r\sin\theta}$;

(ii) $\frac{\partial z}{\partial r} = \frac{1}{y}\cos\theta - \frac{x}{y^2}\sin\theta = \frac{\cos\theta}{r\sin\theta} - \frac{\cos\theta}{r\sin\theta} = 0,$

$\frac{\partial z}{\partial \theta} = \frac{1}{y}(-r\sin\theta) - \frac{x}{y^2}r\cos\theta = -1 - \frac{\cos^2\theta}{\sin^2\theta};$

(iii) Use the Chain Rule in three variables x, y, u: $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial r}$, $\frac{\partial z}{\partial \theta} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial z}{\partial u}\frac{\partial u}{\partial \theta}$:

$\frac{\partial z}{\partial r} = 2x\cos\theta + 2y\sin\theta + 2u \cdot 1 = 2r\cos^2\theta + 2r\sin^2\theta + 2r = 4r,$

$\frac{\partial z}{\partial \theta} = 2x(-r\sin\theta) + 2yr\cos\theta + 2u \cdot 0 = -2r^2\cos\theta\sin\theta + 2r^2\sin\theta\cos\theta = 0.$