Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

S h e e t 20

Due: in the tutorial sessions first Wednesday/Thursday in the next term

Exercise 1

Find the Fourier series of the function $f(x)$:

(i) $f(x) = \begin{cases} 1 \text{ if } -\pi < x < 0 \\ 0 \text{ if } 0 < x < \pi; \end{cases}$

Solution.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2},$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_{-\pi}^{0} \cos nx\,dx = 0,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx = \frac{1}{\pi} \int_{-\pi}^{0} \sin nx\,dx =$$

$$\frac{1}{\pi} \left. \frac{-\cos nx}{n} \right|_{-\pi}^{0} = \frac{-1 + \cos n\pi}{n\pi} = \frac{-1 + (-1)^n}{n\pi},$$

so we have the Fourier Series expansion:

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-1 + (-1)^n}{n\pi} \sin nx.$$

(ii) $f(x) = x$, $-\pi < x < \pi$;

Solution. The function is odd ($f(-x) = -f(x)$), so we only need to calculate b_n:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin nx\,dx = \frac{1}{\pi} \int_{-\pi}^{\pi} u\,dv,$$

where $u = x$ and $dv = \sin nx\,dx$, hence $v = -\frac{\cos nx}{n}$. Thus

$$b_n = \frac{1}{\pi} \left. (uv) \right|_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} v du = \frac{1}{\pi} \left. \frac{-x\cos nx}{n} \right|_{-\pi}^{\pi} - \frac{1}{\pi} \int_{-\pi}^{\pi} -\frac{\cos nx}{n} dx =$$

$$-\frac{2\cos n\pi}{n}+0=\frac{2(-1)^{n+1}}{n},$$

so we have the Fourier Series expansion:

$$f(x)=\sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx.$$

(iii) $f(x)=\begin{cases} x \text{ if } -\pi<x<0 \\ -x \text{ if } 0<x<\pi. \end{cases}$

Solution. Now the function is even ($f(-x)=f(x)$), so we only need to calculate a_n. Furthermore, if $g(x)$ is even, we can simplify our calculation by the formula $\int_{-\pi}^{\pi} g(x)\,dx = 2\int_0^{\pi} g(x)\,dx$. We have

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 2\frac{1}{2\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2}.$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 2\frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx = \ldots$$

The end of calculation is similar to the previous case using integration by parts and is omitted.