Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

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Due: in the tutorial sessions first Wednesday/Thursday in the next term

Exercise 1

Find the characteristic polynomials of the following matrices:

(i) $\begin{pmatrix} 4 & 0 \\ 0 & -3 \end{pmatrix}$;

Solution.

$$p(\lambda) = \det \begin{pmatrix} \lambda - 4 & 0 \\ 0 & \lambda + 3 \end{pmatrix} = (\lambda - 4)(\lambda + 3).$$

(ii) $\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$;

Solution.

$$p(\lambda) = \det \begin{pmatrix} \lambda & -1 \\ -2 & \lambda \end{pmatrix} = \lambda^2 + 2.$$

(iii) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$;

Solution.

$$p(\lambda) = \det \begin{pmatrix} \lambda & -1 & -1 \\ 0 & \lambda - 2 & -2 \\ 0 & 0 & \lambda - 3 \end{pmatrix} = \lambda(\lambda - 2)(\lambda - 3).$$

(iv) $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 3 \end{pmatrix}$.

Solution.

$$p(\lambda) = \det \begin{pmatrix} \lambda & -1 & -1 \\ 0 & \lambda - 2 & -2 \\ 0 & -2 & \lambda - 3 \end{pmatrix} = \lambda[(\lambda-2)(\lambda-3) - (-2)^2].$$

Exercise 2

Find the eigenvalues and the corresponding eigenvectors of the following matrices:

(i) $\begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix}$;

Solution.

$$p(\lambda) = \det \begin{pmatrix} \lambda - 2 & 0 \\ -1 & \lambda + 3 \end{pmatrix} = (\lambda - 2)(\lambda + 3).$$

The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = -3$.

For $\lambda_1 = 2$, the eigenvectors are nonzero solutions of

$$\begin{pmatrix} 2-2 & 0 \\ -1 & 2+3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

and we can put $x_2 = 1$ and get $x_1 = 5$ from the last equation. So $(5, 1)$ is an eigenvector corresponding to $\lambda_1 = 2$.

Similarly, for $\lambda_2 = -3$, we have to solve

$$\begin{pmatrix} -3-2 & 0 \\ -1 & -3+3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0,$$

where we find e.g. $(0, 1)$ as an eigenvector.

(ii) $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 2 & 4 \end{pmatrix}$;

Solution.

$$p(\lambda) = \det \begin{pmatrix} \lambda - 1 & -1 & -2 \\ 0 & \lambda - 2 & -4 \\ 0 & -2 & \lambda - 4 \end{pmatrix} = \lambda(\lambda - 1)(\lambda - 6).$$

The eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = 6$.

For the eigenvectors corresponding to $\lambda_1 = 0$, we have

$$\begin{pmatrix} -1 & -1 & -2 \\ 0 & -2 & -4 \\ 0 & -2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

where a nonzero solution is $(0, -2, 1)$. The calculation of the eigenvectors corresponding to $\lambda_2 = 1$ and $\lambda_3 = 6$ is similar and omitted.

Exercise 3

Find a matrix P that diagonalizes the given matrix A and determine $D = P^{-1}AP$:

(i) $A = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$;

Solution.

$$p(\lambda) = \det \begin{pmatrix} \lambda & -2 \\ -4 & \lambda \end{pmatrix} = \lambda^2 - 8.$$

The eigenvalues are $\lambda_1 = \sqrt{8}$ and $\lambda_2 = -\sqrt{8}$ and as before we can find as eigenvectors correspondingly $\mathbf{v}_1 = (2, \sqrt{8})$ and $\mathbf{v}_2 = (-2, \sqrt{8})$. Then P has $\mathbf{v}_1$ and $\mathbf{v}_2$ as its columns:

$$P = \begin{pmatrix} 2 & -2 \\ \sqrt{8} & \sqrt{8} \end{pmatrix}$$

and D is the diagonal matrix having λ_1 and λ_2 on its diagonal:

$$D = \begin{pmatrix} \sqrt{8} & 0 \\ 0 & -\sqrt{8} \end{pmatrix}.$$

(ii) $A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Solution.

$$p(\lambda) = \det \begin{pmatrix} \lambda - 2 & 0 & 2 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 3 \end{pmatrix} = (\lambda - 2)(\lambda - 3)^2.$$

We have the eigenvalues $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = 3$. For $\lambda_1 = 2$, we find $\mathbf{v}_1 = (1, 0, 0)$ as an eigenvector. For $\lambda_2 = \lambda_3 = 3$, we have the system

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0,$$

for which we have to find 2 linearly independent solution vectors. The system reduces to one equation $x_1 + 2x_3 = 0$, where x_2 and x_3 can be seen as free and $x_1 = -2x_3$. So we have the general solution putting $x_2 = t$, $x_3 = s$ and $x_1 = -2s$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2s \\ t \\ s \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

and we can take vectors $\mathbf{v}_2 = (0, 1, 0)$ and $\mathbf{v}_3 = (-2, 0, 1)$ in front of t and s respectively as linearly independent eigenvectors. Now we obtain P having $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ as columns:

$$P = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and D having λ_1, λ_2, λ_3 on the main diagonal:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

To verify the correctness of calculations, it is recommended to check that

$$PD = AP.$$