Exercise 1

Calculate the length of $u = (1, -1, 1)$, the distance between $u$ and $v = (0, 1, 1)$ and the angle between $u$ and $v$

(i) with respect to the (standard) Euclidean inner product;

Solution.

$$
\|u\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3},
$$

$$
d(u, v) = \sqrt{(1 - 0)^2 + (-1 - 1)^2 + (1 - 1)^2} = \sqrt{5}.
$$

(ii) with respect to the inner product given by $\langle u, v \rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$.

Solution.

$$
\|u\| = \sqrt{2 \cdot 1^2 + 3(-1)^2 + 1^2} = \sqrt{6},
$$

$$
d(u, v) = \sqrt{2(1 - 0)^2 + 3(-1 - 1)^2 + (1 - 1)^2} = \sqrt{14}.
$$

Exercise 2

Which of the following bases are orthogonal and which are orthonormal?

(i) $(1, 0), (0, 2)$;

Solution. We have

$$
\langle (1, 0), (0, 2) \rangle = 1 \cdot 0 + 0 \cdot 2 = 0,
$$

hence an orthogonal basis. Furthermore $\| (0, 2) \| = \sqrt{2} \neq 1$, hence not an orthonormal basis.
(ii) \((1, 0, 1), (1, 1, -1), (-1, 0, 1)\);

**Solution.** We have \(\langle (1, 1, -1), (-1, 0, 1) \rangle = -2 \neq 0\), hence not an orthogonal and not an orthonormal basis.

(iii) \((1, 0, 0), (0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\);

**Solution.** We have
\[
\langle (1, 0, 0), (0, 1/\sqrt{2}, -1/\sqrt{2}) \rangle = \langle (1, 0, 0), (0, 1/\sqrt{2}, 1/\sqrt{2}) \rangle = 0
\]
\[
\langle (0, 1/\sqrt{2}, -1/\sqrt{2}), (0, 1/\sqrt{2}, 1/\sqrt{2}) \rangle = 0,
\]
hence an orthogonal basis. Furthermore,
\[
\| (1, 0, 0) \| = \| (0, 1/\sqrt{2}, -1/\sqrt{2}) \| = \| (0, 1/\sqrt{2}, 1/\sqrt{2}) \| = 1,
\]
hence an orthonormal basis.

**Exercise 3**

Calculate the coordinates of \(\mathbf{v}\) relative to the basis in Exercise 2 (iii):

(i) \(\mathbf{v} = (1, 1, 1)\);

**Solution.** The basis is orthonormal, so the coordinates can be calculated as products:
\[
c_1 = \langle \mathbf{v}, \mathbf{v}_1 \rangle = \langle (1, 1, 1), (1, 0, 0) \rangle = 1,
\]
\[
c_2 = \langle (1, 1, 1), (0, 1/\sqrt{2}, -1/\sqrt{2}) \rangle = 0,
\]
\[
c_3 = \langle (1, 1, 1), (0, 1/\sqrt{2}, 1/\sqrt{2}) \rangle = 2/\sqrt{2}.
\]

(ii) \(\mathbf{v} = (-1, 1, -1)\).

**Solution.** Solution is similar.

**Exercise 4**

Use the Gram-Schmidt process to transform \(\mathbf{u}_1 = (1, 1, 1), \mathbf{u}_2 = (1, 1, 0), \mathbf{u}_3 = (1, 0, 0)\) into an orthogonal basis.
Solution. Following the process, we find:

\[ v_1 = u_1 = (1, 1, 1), \]

\[ v_2 = u_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} v_1 = (1, 1, 0) - \frac{2}{3} (1, 1, 1) = \left( \frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right), \]

\[ v_3 = u_3 - \frac{\langle u_3, v_1 \rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2 \rangle}{\|v_2\|^2} v_2 = (1, 0, 0) - \frac{1}{3} (1, 1, 1) - \frac{1}{6/9} \left( \frac{1}{3}, \frac{1}{3}, -\frac{2}{3} \right) = \ldots \]