Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

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Due: in the tutorial sessions first Wednesday/Thursday in the next term

Exercise 1

Calculate the length of $\mathbf{u} = (1, -1, 1)$, the distance between $\mathbf{u}$ and $\mathbf{v} = (0, 1, 1)$ and the angle between $\mathbf{u}$ and $\mathbf{v}$

(i) with respect to the (standard) Euclidean inner product;

Solution.

$$\|\mathbf{u}\| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{3},$$

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{(1-0)^2 + (-1-1)^2 + (1-1)^2} = \sqrt{5}.$$

(ii) with respect to the inner product given by $\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$.

Solution.

$$\|\mathbf{u}\| = \sqrt{2 \cdot 1^2 + 3(-1)^2 + 1^2} = \sqrt{6},$$

$$d(\mathbf{u}, \mathbf{v}) = \sqrt{2(1-0)^2 + 3(-1-1)^2 + (1-1)^2} = \sqrt{14}.$$

Exercise 2

Which of the following bases are orthogonal and which are orthonormal?

(i) $(1, 0)$, $(0, 2)$;

Solution. We have

$$\langle (1, 0), (0, 2) \rangle = 1 \cdot 0 + 0 \cdot 2 = 0,$$

hence an orthogonal basis. Furthermore $\|(0, 2)\| = \sqrt{2} \neq 1$, hence not an orthonormal basis.

(ii) $(1,0,1)$, $(1,1,-1)$, $(-1,0,1)$;

Solution. We have $\langle(1,1,-1),(-1,0,1)\rangle = -2 \neq 0$, hence not an orthogonal and not an orthonormal basis.

(iii) $(1,0,0)$, $(0,\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})$, $(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$;

Solution. We have

$$\langle(1,0,0),(0,1/\sqrt{2},-1/\sqrt{2})\rangle = \langle(1,0,0),(0,1/\sqrt{2},1/\sqrt{2})\rangle =$$

$$\langle(0,1/\sqrt{2},-1/\sqrt{2}),(0,1/\sqrt{2},1/\sqrt{2})\rangle = 0,$$

hence an orthogonal basis. Furthermore,

$$\|(1,0,0)\| = \|(0,1/\sqrt{2},-1/\sqrt{2})\| = \|(0,1/\sqrt{2},1/\sqrt{2})\| = 1,$$

hence an orthonormal basis.

Exercise 3

Calculate the coordinates of $\mathbf{v}$ relative to the basis in Exercise 2 (iii):

(i) $\mathbf{v} = (1,1,1)$;

Solution. The basis is orthonormal, so the coordinates can be calculated as products:

$$c_1 = \langle\mathbf{v},\mathbf{v_1}\rangle = \langle(1,1,1),(1,0,0)\rangle = 1,$$

$$c_2 = \langle(1,1,1),(0,1/\sqrt{2},-1/\sqrt{2})\rangle = 0,$$

$$c_3 = \langle(1,1,1),(0,1/\sqrt{2},1/\sqrt{2})\rangle = 2/\sqrt{2}.$$

(ii) $\mathbf{v} = (-1,1,-1)$.

Solution. Solution is similar.

Exercise 4

Use the Gram-Schmidt process to transform $\mathbf{u}_1 = (1,1,1)$, $\mathbf{u}_2 = (1,1,0)$, $\mathbf{u}_3 = (1,0,0)$ into an orthogonal basis.

Solution. Following the process, we find:

$$\mathbf{v}_1 = \mathbf{u}_1 = (1,1,1),$$

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 = (1,1,0) - \frac{2}{3}(1,1,1) = \left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right),$$

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\|\mathbf{v}_1\|^2}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\|\mathbf{v}_2\|^2}\mathbf{v}_2 = (1,0,0) - \frac{1}{3}(1,1,1) - \frac{1/3}{6/9}\left(\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}\right) = \ldots$$