Exercise 1
Find the coordinates of the vector $v$ with respect to the basis $v_1, \ldots, v_n$ (i.e. the coefficients $c_1, \ldots, c_n$ in the representation $v = c_1v_1 + \cdots + c_nv_n$):

(i) $v = (3, -7), v_1 = (1, -1), v_2 = (1, 1)$;

**Solution.** We have to solve the vector equation $v = c_1v_1 + c_2v_2$ or the system

$$
\begin{align*}
3 &= c_1 + c_2 \\
-7 &= c_1 - c_2.
\end{align*}
$$

(ii) $v = (2, -1, 3), v_1 = (1, 0, 0), v_2 = (1, 1, 0), v_3 = (1, 1, 1)$;

**Solution.** Here the answer comes from solving the system

$$
\begin{align*}
2 &= c_1 + c_2 + c_3 \\
-1 &= c_2 + c_3 \\
3 &= c_3.
\end{align*}
$$

(iii) $v = (1, 1, 1, 1), v_1 = (1, 0, 1, 0), v_2 = (1, 1, 0, 0), v_3 = (0, 1, 1, 0), v_4 = (-1, 1, 1, 1)$.

**Solution.** And here the system is

$$
\begin{align*}
1 &= c_1 + c_2 - c_4 \\
1 &= c_2 + c_3 + c_4 \\
1 &= c_1 + c_3 + c_4 \\
1 &= c_4.
\end{align*}
$$
Exercise 2

Find the vector form $\mathbf{x} = \mathbf{x}_0 + c_1 \mathbf{v}_1 + \cdots + c_n \mathbf{v}_n$ (i.e. find vectors $\mathbf{x}_0, \mathbf{v}_1, \ldots, \mathbf{v}_n$) for the general solution of the system:

(i)

\[
\begin{aligned}
&x_1 - 3x_2 + x_3 = 1 \\
&x_2 = 2
\end{aligned}
\]

**Solution.** We have $x_2 = 2$ from the 2nd equation and, substituting into the 1st, $x_1 = 7 - x_3$. So $x_3 = t$ is a free parameter and $x_1 = 7 - t$, $x_2 = 2$:

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 - t \\ 2 \\ t \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}
\]

and we find \( \mathbf{x} = \mathbf{x}_0 + c_1 \mathbf{v}_1 = (7, 2, 0) + c_1 (-1, 0, 1) \).

(ii)

\[
\begin{aligned}
x_1 + x_2 + 2x_3 &= 5 \\
x_1 + x_3 - x_4 &= -2 \\
2x_1 + x_2 + 3x_3 - x_4 &= 3
\end{aligned}
\]

**Solution.** Bigger systems are more convenient to solve using the method of elementary row operations. We have the matrix

\[
\begin{pmatrix}
1 & 1 & 2 & 0 & | & 5 \\
1 & 0 & 1 & -1 & | & -2 \\
2 & 1 & 3 & -1 & | & 3
\end{pmatrix}
\]

which we want to bring to a row echelon form. We use the leading 1 of the 1st row to eliminate the leading terms of the other rows. So we subtract the 1st row from the 2nd and the 1st times 2 from the 3rd:

\[
\begin{pmatrix}
1 & 1 & 2 & 0 & | & 5 \\
0 & -1 & -1 & -1 & | & -7 \\
0 & -1 & -1 & -1 & | & -7
\end{pmatrix}
\]

Now to get the leading 1, multiply the 2nd row by 1 and then add it to the 3rd row:

\[
\begin{pmatrix}
1 & 1 & 2 & 0 & | & 5 \\
0 & 1 & 1 & 1 & | & 7 \\
0 & 0 & 0 & 0 & | & 0
\end{pmatrix}
\]
We can further improve the matrix by using the leading 1 of the 2nd row to eliminate all the other entries in the 2nd column. So we subtract the 2nd row from the 1st:

\[
\begin{pmatrix}
1 & 0 & 1 & -1 & -2 \\
0 & 1 & 1 & 1 & 7 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Now we can go back to the system corresponding to this matrix:

\[
\begin{aligned}
x_1 + x_3 - x_4 &= -2 \\
x_2 + x_3 + x_4 &= 7,
\end{aligned}
\]

where we have the free variables \(x_3 = t, x_4 = s\) and solve for the others \(x_1 = -2 - t + s, x_2 = 7 - t - s\):

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{pmatrix} = \begin{pmatrix}
-2 - t + s \\
7 - t - s \\
t \\
s
\end{pmatrix} + \begin{pmatrix}
-1 \\
0 \\
1 \\
0
\end{pmatrix} + \begin{pmatrix}
1 \\
0 \\
0 \\
1
\end{pmatrix}
\]

Exercise 3

Find a basis for the nullspace of \(A\):

(i)

\[
A = \begin{pmatrix}
1 & 1 & 1 \\
1 & -1 & 1
\end{pmatrix}
\]

**Solution.** Using elementary row operations, we obtain the row equivalent matrices

\[
A \sim \begin{pmatrix}
1 & 1 & 1 \\
0 & -2 & 0
\end{pmatrix} \sim \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 0
\end{pmatrix} \sim \begin{pmatrix}
1 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix},
\]

where the system corresponding to the last matrix can be solved as

\[
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} = t \begin{pmatrix}
-1 \\
0 \\
1
\end{pmatrix}
\]

and hence the basis of the nullspace is \(\{(-1, 0, 1)\}\) (a set of only one vector).
(ii) \[ A = \begin{pmatrix}
  2 & 0 & -1 & 0 \\
  1 & -1 & 1 & -1 \\
  3 & -1 & 0 & -1 \\
  1 & 1 & -2 & 1 \\
\end{pmatrix} \]

**Solution.** Again we use elementary row operations. Exchange the 1st and the 2nd rows to obtain the leading 1 for the first row:

\[ \begin{pmatrix}
  1 & -1 & 1 & -1 \\
  2 & 0 & -1 & 0 \\
  3 & -1 & 0 & -1 \\
  1 & 1 & -2 & 1 \\
\end{pmatrix} \]

Now subtract the 1st row times 2 from the 2nd, times 3 from the 3rd and without any factor from the 4th row:

\[ \begin{pmatrix}
  1 & -1 & 1 & -1 \\
  0 & 2 & -3 & 2 \\
  0 & 2 & -3 & 2 \\
  0 & 2 & -3 & 2 \\
\end{pmatrix} \]

Now clearly the 3rd and the 4th rows are fully eliminated, so we can omit them. We obtain the system

\[ \begin{align*}
  x_1 - x_2 + x_3 - x_4 &= 0 \\
  2x_2 - 3x_3 + 2x_4 &= 0
\end{align*} \]

that we can solve as

\[ \begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3 \\
  x_4
\end{pmatrix} = \begin{pmatrix}
  \frac{1}{2}t \\
  \frac{3}{2}t - s \\
  t \\
  s
\end{pmatrix} = t \begin{pmatrix}
  1/2 \\
  3/2 \\
  1 \\
  0
\end{pmatrix} + s \begin{pmatrix}
  0 \\
  -1 \\
  1 \\
  1
\end{pmatrix}, \]

so the basis is \{ (1/2, 3/2, 1, 0), (0, −1, 0, 1) \}. 