March 1, 2005 Lecturer Dmitri Zaitsev Hilary Term 2005

Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

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Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Find the coordinates of the vector $\mathbf{v}$ with respect to the basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (i.e. the coefficients $c_1, \ldots, c_n$ in the representation $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$:

(i) $\mathbf{v} = (3, -7)$, $\mathbf{v}_1 = (1, -1)$, $\mathbf{v}_2 = (1, 1)$;

Solution. We have to solve the vector equation $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$ or the system

$$\begin{cases} 3 = c_1 + c_2 \\ -7 = c_1 - c_2. \end{cases}$$

(ii) $\mathbf{v} = (2, -1, 3)$, $\mathbf{v}_1 = (1, 0, 0)$, $\mathbf{v}_2 = (1, 1, 0)$, $\mathbf{v}_3 = (1, 1, 1)$;

Solution. Here the answer comes from solving the system

$$\begin{cases} 2 = c_1 + c_2 + c_3 \\ -1 = c_2 + c_3 \\ 3 = c_3. \end{cases}$$

(iii) $\mathbf{v} = (1, 1, 1, 1)$, $\mathbf{v}_1 = (1, 0, 1, 0)$, $\mathbf{v}_2 = (1, 1, 0, 0)$, $\mathbf{v}_3 = (0, 1, 1, 0)$, $\mathbf{v}_4 = (-1, 1, 1, 1)$.

Solution. And here the system is

$$\begin{cases} 1 = c_1 + c_2 - c_4 \\ 1 = c_2 + c_3 + c_4 \\ 1 = c_1 + c_3 + c_4 \\ 1 = c_4. \end{cases}$$

Exercise 2

Find the vector form $\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$ (i.e. find vectors $\mathbf{x}_0$, $\mathbf{v}_1, \ldots, \mathbf{v}_n$) for the general solution solution of the system:

(i)

$$\begin{cases} x_1 - 3x_2 + x_3 = 1 \\ x_2 = 2 \end{cases}$$

Solution. We have $x_2 = 2$ from the 2nd equation and, substituting into the 1st, $x_1 = 7 - x_3$. So $x_3 = t$ is a free parameter and $x_1 = 7 - t$, $x_2 = 2$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7-t \\ 2 \\ t \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

and we find $\mathbf{x} = \mathbf{x}_0 + c_1\mathbf{v}_1 = (7, 2, 0) + c_1(-1, 0, 1)$.

(ii)

$$\begin{cases} x_1 + x_2 + 2x_3 = 5 \\ x_1 + x_3 - x_4 = -2 \\ 2x_1 + x_2 + 3x_3 - x_4 = 3 \end{cases}$$

Solution. Bigger systems are more convenient to solve using the method of elementary row operations. We have the matrix

$$\begin{pmatrix} 1 & 1 & 2 & 0 & | & 5 \\ 1 & 0 & 1 & -1 & | & -2 \\ 2 & 1 & 3 & -1 & | & 3 \end{pmatrix}$$

which we want to bring to a row echelon form. We use the leading 1 of the 1st row to eliminate the leading terms of the other rows. So we subtract the 1st row from the 2nd and the 1st times 2 from the 3rd:

$$\begin{pmatrix} 1 & 1 & 2 & 0 & | & 5 \\ 0 & -1 & -1 & -1 & | & -7 \\ 0 & -1 & -1 & -1 & | & -7 \end{pmatrix}$$

Now to get the leading 1, multiply the 2nd row by 1 and then add it to the 3rd row:

$$\begin{pmatrix} 1 & 1 & 2 & 0 & | & 5 \\ 0 & 1 & 1 & 1 & | & 7 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

We can further improve the matrix by using the leading 1 of the 2nd row to eliminate all the other entries in the 2nd column. So we subtract the 2nd row from the 1st:

$$\begin{pmatrix} 1 & 0 & 1 & -1 & | & -2 \\ 0 & 1 & 1 & 1 & | & 7 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

Now we can go back to the system corresponding to this matrix:

$$\begin{cases} x_1 + x_3 - x_4 = -2 \\ x_2 + x_3 + x_4 = 7, \end{cases}$$

where we have the free variables $x_3 = t$, $x_4 = s$ and solve for the others $x_1 = -2 - t + s$, $x_2 = 7 - t - s$:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 - t + s \\ 7 - t - s \\ t \\ s \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Exercise 3

Find a basis for the nullspace of A:

(i)

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}$$

Solution. Using elementary row operations, we obtain the row equivalent matrices

$$A \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where the system corresponding to the last matrix can be solved as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

and hence the basis of the nullspace is $\{(-1, 0, 1)\}$ (a set of only one vector).

(ii)

$$A = \begin{pmatrix} 2 & 0 & -1 & 0 \\ 1 & -1 & 1 & -1 \\ 3 & -1 & 0 & -1 \\ 1 & 1 & -2 & 1 \end{pmatrix}$$

Solution. Again we use elementary row operations. Exchange the 1st and the 2nd rows to obtain the leading 1 for the first row:

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 2 & 0 & -1 & 0 \\ 3 & -1 & 0 & -1 \\ 1 & 1 & -2 & 1 \end{pmatrix}.$$

Now subtract the 1st row times 2 from the 2nd, times 3 from the 3rd and without any factor from the 4th row:

$$\begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 2 & -3 & 2 \\ 0 & 2 & -3 & 2 \\ 0 & 2 & -3 & 2 \end{pmatrix}.$$

Now clearly the 3rd and the 4th rows are fully eliminated, so we can omit them. We obtain the system

$$\begin{cases} x_1 - x_2 + x_3 - x_4 = 0 \\ 2x_2 - 3x_3 + 2x_4 = 0 \end{cases}$$

that we can solve as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}t \\ \frac{3}{2}t - s \\ t \\ s \end{pmatrix} = t \begin{pmatrix} 1/2 \\ 3/2 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},$$

so the basis is $\{(1/2, 3/2, 1, 0), (0, -1, 0, 1)\}$.