February 15, 2005 Lecturer Dmitri Zaitsev Hilary Term 2005

Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

S h e e t 14

Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Determine which of the following are subspaces of $\mathbb{R}^3$:

(i) the set of all vectors of the form $(0, 0, a)$;

Solution. Given any two vectors in this set $\mathbf{v}_1 = (0, 0, a_1)$ and $\mathbf{v}_2 = (0, 0, a_2)$, their sum is $\mathbf{v}_1 + \mathbf{v}_2 = (0, 0, a_1 + a_2)$ which is again in the set. Give any vector $\mathbf{v} = (0, 0, a)$ in the set and a scalar k, the product $k\mathbf{v} = (0, 0, ka)$ is again in the set. Thus the set is a subspace.

(ii) the set of all vectors of the form $(0, 1, a)$;

Solution. Given any two vectors in this set $\mathbf{v}_1 = (0, 1, a_1)$ and $\mathbf{v}_2 = (0, 1, a_2)$, their sum is $\mathbf{v}_1 + \mathbf{v}_2 = (0, 2, a_1 + a_2)$ which is not in the set. Thus the set is not a subspace.

(iii) the set of all vectors of the form (a, b, c), where $a + b = c$;

(iv) the set of all solutions (x, y, z) of the system $x + 2y = 0$, $z - 4y = 0$.

Solution. Both sets are sets of solutions of systems of linear homogeneous equations. Hence they are subspaces.

Exercise 2

Determine whether the vectors span $\mathbb{R}^3$:

(i) $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (2, 0, 1)$, $\mathbf{v}_3 = (1, 0, 0)$;

Solution. The vectors span $\mathbb{R}^3$ if any vector $\mathbf{v} = (a, b, c)$ can be represented as a linear combination $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$. To check it, we have to solve for k_1, k_2, k_3 the vector equation $\mathbf{v} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3$ that is equivalent to the system

$$\begin{cases} a = k_1 + 2k_2 + 3k_3 \\ b = 0k_1 + 0k_2 + 0k_3 \\ c = k_1 + 0k_2 + 0k_3. \end{cases}$$

It is clear that the 2nd equation is only solvable for $b = 0$, hence not for all vectors $\mathbf{v}$. Thus the vectors do not span $\mathbb{R}^3$.

(ii) $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (2, 0, 1)$, $\mathbf{v}_3 = (1, 0, 0)$, $\mathbf{v}_4 = (1, 1, 0)$.

Solution. Proceeding as before we come to the system (note that we have 4 vectors so we need 4 scalars k_1, k_2, k_3, k_4 for their linear combination):

$$\begin{cases} a = k_1 + 2k_2 + 3k_3 + k_4 \\ b = 0k_1 + 0k_2 + 0k_3 + k_4 \\ c = k_1 + k_2 + 0k_3 + 0k_4 \end{cases} \text{or} \begin{cases} a = k_1 + 2k_2 + 3k_3 + k_4 \\ b = k_4 \\ c = k_1 + k_2. \end{cases}$$

Now, for any choice of a, b, c, we can find a solution (k_1, k_2, k_3, k_4). Hence any $\mathbf{v}$ can be written as a linear combination $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + k_4\mathbf{v}_4$ and therefore the vectors span $\mathbb{R}^3$.

Determine whether the vectors span $\mathbb{R}^4$:
(iii) $\mathbf{v}_1 = (1, 0, 1, 1)$, $\mathbf{v}_2 = (2, 0, 1, 0)$, $\mathbf{v}_3 = (1, 0, 0, 0)$, $\mathbf{v}_4 = (1, 1, 0, 0)$.

Solution. Now we have 4 vectors in $\mathbb{R}^4$, so we have a system of 4 equation with 4 unknown scalars:

$$\begin{cases} a = k_1 + 2k_2 + 3k_3 + k_4 \\ b = k_4 \\ c = k_1 + k_2. \\ d = k_1. \end{cases}$$

Again, the system can be always solved, hence the vectors span $\mathbb{R}^4$.

Exercise 3

Find parametric equations for the line spanned by the vector:
(i) $\mathbf{u} = (1, 0, 3)$;

Solution.

$$x = t, \quad y = 0, \quad z = 3t.$$

(ii) $\mathbf{u} = (1, 0, 3, 0, 5)$;

Solution.

$$x_1 = t, \quad x_2 = 0, \quad x_3 = 3t, \quad x_4 = 0, \quad x_5 = 5t.$$

The choice of the variables is arbitrary (but the number is important).

Find an equation for the plane spanned by the vectors:
(iii) $\mathbf{u} = (1, 0, 3)$, $\mathbf{v} = (-1, 0, 3)$;

Solution. A general plane in $\mathbb{R}^3$ is given by an equation $ax+by+cz = 0$ where the coefficients a, b, c are to be determined. The plane will contain the given vectors $(1, 0, 3)$ and $(-1, 0, 3)$ if the equation is satisfied after each of the substitutions $(x, y, z) = (1, 0, 3)$ and $(x, y, z) = (-1, 0, 3)$. Thus we obtain a system

$$\begin{cases} a + 3c = 0 \\ -a + 3c = 0, \end{cases}$$

from where we have $a = c = 0$ and b is free. We can choose e.g. $b = 1$. Then the equation $ax + by + cz = 0$ becomes $y = 0$ which is a desired equation.