Exercise 1

Determine which of the following are subspaces of $\mathbb{R}^3$:

(i) the set of all vectors of the form $(0, 0, a)$;

**Solution.** Given any two vectors in this set $v_1 = (0, 0, a_1)$ and $v_2 = (0, 0, a_2)$, their sum is $v_1 + v_2 = (0, 0, a_1 + a_2)$ which is again in the set. Give any vector $v = (0, 0, a)$ in the set and a scalar $k$, the product $kv = (0, 0, ka)$ is again in the set. Thus the set is a subspace.

(ii) the set of all vectors of the form $(0, 1, a)$;

**Solution.** Given any two vectors in this set $v_1 = (0, 1, a_1)$ and $v_2 = (0, 1, a_2)$, their sum is $v_1 + v_2 = (0, 2, a_1 + a_2)$ which is not in the set. Thus the set is not a subspace.

(iii) the set of all vectors of the form $(a, b, c)$, where $a + b = c$;

(iv) the set of all solutions $(x, y, z)$ of the system $x + 2y = 0$, $z - 4y = 0$.

**Solution.** Both sets are sets of solutions of systems of linear homogeneous equations. Hence they are subspaces.

Exercise 2

Determine whether the vectors span $\mathbb{R}^3$:

(i) $v_1 = (1, 0, 1)$, $v_2 = (2, 0, 1)$, $v_3 = (1, 0, 0)$;
Solution. The vectors span $\mathbb{R}^3$ if any vector $v = (a, b, c)$ can be represented as a linear combination $k_1v_1 + k_2v_2 + k_3v_3$. To check it, we have to solve for $k_1, k_2, k_3$ the vector equation $v = k_1v_1 + k_2v_2 + k_3v_3$ that is equivalent to the system

\[
\begin{align*}
a &= k_1 + 2k_2 + 3k_3 \\
b &= 0k_1 + 0k_2 + 0k_3 \\
c &= k_1 + 0k_2 + 0k_3.
\end{align*}
\]

It is clear that the 2nd equation is only solvable for $b = 0$, hence not for all vectors $v$. Thus the vectors do not span $\mathbb{R}^3$.

(ii) $v_1 = (1, 0, 1), v_2 = (2, 0, 1), v_3 = (1, 0, 0), v_4 = (1, 1, 0)$.

Solution. Proceeding as before we come to the system (note that we have 4 vectors so we need 4 scalars $k_1, k_2, k_3, k_4$ for their linear combination):

\[
\begin{align*}
a &= k_1 + 2k_2 + 3k_3 + k_4 \\
b &= 0k_1 + 0k_2 + 0k_3 + k_4 \\
c &= k_1 + k_2 + 0k_3 + 0k_4 \\
d &= k_1 + k_2.
\end{align*}
\]

Now, for any choice of $a, b, c, d$, we can find a solution $(k_1, k_2, k_3, k_4)$. Hence any $v$ can be written as a linear combination $k_1v_1 + k_2v_2 + k_3v_3 + k_4v_4$ and therefore the vectors span $\mathbb{R}^3$.

Determine whether the vectors span $\mathbb{R}^4$:

(iii) $v_1 = (1, 0, 1, 1), v_2 = (2, 0, 1, 0), v_3 = (1, 0, 0, 0), v_4 = (1, 1, 0, 0)$.

Solution. Now we have 4 vectors in $\mathbb{R}^4$, so we have a system of 4 equation with 4 unknown scalars:

\[
\begin{align*}
a &= k_1 + 2k_2 + 3k_3 + k_4 \\
b &= k_4 \\
c &= k_1 + k_2. \\
d &= k_1.
\end{align*}
\]

Again, the system can be always solved, hence the vectors span $\mathbb{R}^4$.

Exercise 3

Find parametric equations for the line spanned by the vector:

(i) $u = (1, 0, 3)$;
Solution.

\[ x = t, \quad y = 0, \quad z = 3t. \]

(ii) \( u = (1, 0, 3, 0, 5); \)

Solution.

\[ x_1 = t, \quad x_2 = 0, \quad x_3 = 3t, \quad x_4 = 0, \quad x_5 = 5t. \]

The choice of the variables is arbitrary (but the number is important).

Find an equation for the plane spanned by the vectors:

(iii) \( u = (1, 0, 3), \ v = (-1, 0, 3); \)

Solution. A general plane in \( \mathbb{R}^3 \) is given by an equation \( ax + by + cz = 0 \) where the coefficients \( a, b, c \) are to be determined. The plane will contain the given vectors \((1, 0, 3)\) and \((-1, 0, 3)\) if the equation is satisfied after each of the substitutions \((x, y, z) = (1, 0, 3)\) and \((x, y, z) = (-1, 0, 3)\). Thus we obtain a system

\[
\begin{cases}
a + 3c = 0 \\
-a + 3c = 0,
\end{cases}
\]

from where we have \( a = c = 0 \) and \( b \) is free. We can choose e.g. \( b = 1 \). Then the equation \( ax + by + cz = 0 \) becomes \( y = 0 \) which is a desired equation.