February 8, 2005 Lecturer Dmitri Zaitsev Hilary Term 2005

Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

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Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Find the standard matrix for the linear transformations T defined by the equations

(i) $w_1 = x_1 - x_2, \quad w_2 = x_1 + x_2,$

Solution.

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

(ii) $w_1 = 2x - z, \quad w_2 = y, \quad w_3 = z,$

Solution.

$$\begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(iii) $w_1 = x_1, \quad w_2 = x_1 + x_2, \quad w_3 = x_1 + x_2 + x_3, \quad w_4 = x_1 + x_2 + x_3 + x_4,$

Solution.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

and by the formulas

(v) $T(x_1, x_2) = (x_1, -x_2),$

Solution.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(iv) $T(x_1, x_2, x_3) = (x_2, -x_1, x_2 + x_1, 3x_3, -4x_3)$.

Solution.

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -4 \end{pmatrix}$$

Exercise 2

Find $T(\mathbf{x}) = A\mathbf{x}$ for the matrix A and the vector $\mathbf{x}$ whenever the product makes sense (i.e. the sizes of A and $\mathbf{x}$ fit together):

(i) $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$,

Solution.

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

(ii) $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$,

Solution. 2x3 and 2x1 matrices cannot be multiplied.

(iii) $A = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

Solution.

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Exercise 3

Use matrix multiplication to find:

(i) the reflection of the vector $(1, -2)$ about the y-axis;

Solution. The matrix of the reflection about the y-axis is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, the result is

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

(ii) the orthogonal projection of the vector $(-1, 2)$ to the x-axis;

Solution. The matrix of the orthogonal projection to the x-axis is $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, the result is

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

(iii) the image of the vector $(-1, 1)$ under rotation through the angle $\frac{\pi}{3}$ about the origin.

Solution. The matrix of the rotation through the angle $\frac{\pi}{3}$ about the origin is

$$\begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix},$$

the result is

$$\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

(iv) the image of the vector $(-2, 7)$ under rotation through the angle $-\frac{\pi}{4}$ about the origin.

Solution.

$$\begin{pmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{pmatrix} \begin{pmatrix} -2 \\ 7 \end{pmatrix}.$$