Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

Sheet 12

Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Find the absolute maximum and minimum of the function f on the region R (use the first derivative test to find the critical points in the interior, the method of Lagrange multipliers for the boundary curves, add the corners to the set of points found and compare the values of f to select the maximal and minimal ones):

(i) $f(x,y) = x^2 + y^2 - 2x - 2y$, R is the triangular bounded by the x- and y-axes and the line x + y = 3;

Solution. The first derivative test $f_x = f_y = 0$ gives the point (1, 1) which is in the interior of the region, hence is of interest. Next consider the boundary curves that are the 3 lines x = 0, y = 0 and x + y = 3.

The curve x = 0 can be written as g(x, y) = 0 with g = x. The Lagrange multiplier method gives the system

$$\begin{cases} \nabla f = \lambda \nabla g, \\ g = 0, \end{cases}$$

which takes the form

$$\begin{cases} 2x - 2 = \lambda, \\ 2y - 2 = 0, \\ x = 0, \end{cases}$$

with the unique solution (x, y) = (0, 1) which belongs to the boundary of the region.

The curve y = 0 can be written as g(x, y) = 0 with g = y. The Lagrange multiplier method gives now the system

$$\begin{cases} 2x - 2 = 0, \\ 2y - 2 = \lambda, \\ y = 0, \end{cases}$$

with the unique solution (x, y) = (1, 0) which belongs to the boundary of the region.

The curve x + y = 3 can be written as g(x, y) = 0 with g = x + y - 3. The Lagrange multiplier method gives now the system

$$\begin{cases} 2x - 2 = \lambda, \\ 2y - 2 = \lambda, \\ x + y = 3, \end{cases}$$

whose unique solution (x, y) = (3/2, 3/2) again belongs to the boundary of the given region.

Collecting all the solutions, adding the corners (the vertices of the triangle) and calculating the values of f, we obtain:

$$\begin{array}{cccc} (x,y) & f(x,y) \\ (1,1) & -2 \\ (0,1) & -1 \\ (1,0) & -1 \\ (3/2,3/2) & -3/2 \\ (0,3) & 3 \\ (3,0) & 3 \\ (0,0) & 0. \end{array}$$

Hence the absolute minimum of f is -2 and the absolute maximum is 3.

(ii) f(x,y) = x - y, R is the region above the x-axis and below the parabola $y = 4 - x^2$.

Solution. The first derivative test $f_x = f_y = 0$ gives no solutions. Next consider the boundary curves that are the line y = 0 and the parabola $y + x^2 - 4 = 0$. For the line we have the system

$$\begin{cases} 1=0\\ -1=\lambda\\ x=0, \end{cases}$$

with no solutions. For the parabola, we have

$$\begin{cases} 1=2x\lambda\\ -1=\lambda\\ y+x^2-4=0, \end{cases}$$

with the unique solution (x, y) = (-1/2, 15/4). Finally the vertices, where the line and the parabola interesect, are $(\pm 2, 0)$. Calculate the values of f:

$$\begin{array}{rrrr} (x,y) & f(x,y) \\ (-1/2,15/4) & -17/4 \\ (2,0) & 2 \\ (-2,0) & -2. \end{array}$$

Thus -17/4 is the absolute minimum and 2 is the absolute maximum.

Exercise 2

Find $\mathbf{u} - \mathbf{v}$, $5\mathbf{u}$, $\|\mathbf{v}\|$, the dot product $\mathbf{u} \cdot \mathbf{v}$ and determine whether \mathbf{u} and \mathbf{v} are orthogonal (or for which values of parameters, if any are present):

- (i) $\mathbf{u} = (1, 2), \mathbf{v} = (2, -1);$
- (ii) $\mathbf{u} = (1, 0, 1, 0), \mathbf{v} = (1, 1, 1, 1);$
- (iii) $\mathbf{u} = (k, 0, 0, -k), \mathbf{v} = (k, 2, 1, 2);$
- (iv) $\mathbf{u} = (a, b), \mathbf{v} = (-b, 2a).$

Solution. Solution is simple and omitted.