Exercise 1

Write $\iiint_D f(x, y, z) \, dV$ as iterated integral in cylindrical coordinates without evaluating (i.e. write it as iterated integral of a function in $r, \theta, z$ and set up the limits):

(i) $f(x, y, z) = 5$, $D$ is the cylinder $x^2 + y^2 < 1$, $-1 \leq z \leq 1$;

Solution. Substituting the expressions $x = r\cos\theta$, $y = r\sin\theta$ for cylindrical coordinates, we have $0 \leq r \leq 1$ and no restrictions on $\theta$, i.e. $0 \leq \theta \leq 2\pi$.

$$\iiint_D f(x, y, z) \, dV = \int_0^{2\pi} \int_0^1 \int_{-1}^1 5r \, dz \, dr \, d\theta.$$ 

(ii) $f(x, y, z) = x^2 + y^2$, $D$ is the circular cylinder whose base is the circle $(x-1)^2 + y^2 = 1$ in the $xy$-plane and whose top lies in the plane $z = 2 + y$.

Solution. Substituting the expressions $x = r\cos\theta$, $y = r\sin\theta$ for cylindrical coordinates into the equation $(x-1)^2 + y^2 = 1$, we obtain, after cancellation, $r = 2\cos\theta$. Thus, in the region $D$, we have $0 \leq r \leq 2\cos\theta$. This also put the restriction $\cos\theta \geq 0$ which means $-\pi/2 \leq \theta \leq \pi/2$. The additional equations $0 \leq z \leq 2 + y$ put no further restrictions on $r$ and $\theta$.

$$\iiint_D f(x, y, z) \, dV = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^{2+r\sin\theta} r^2 \cdot r \, dz \, dr \, d\theta.$$ 

Exercise 2

Set up the iterated integral with correct limits that calculates the volume of the given solid $D$ in spherical coordinates $\rho, \varphi, \theta$ without evaluating:

(i) $D$ is the solid between the spheres $\rho = 1$ and $\rho = 2$;
Solution. We have the full ranges $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi$ for $\theta$ and $\varphi$ and $1 \leq \rho \leq 2$ between the spheres.

$$V = \int_0^{2\pi} \int_0^\pi \int_1^2 \rho^2 \sin \theta \, d\rho \, d\varphi \, d\theta.$$ 

(ii) $D$ is the solid bounded by the sphere $\rho = 1$ in the half-space $z \geq 0$;

Solution. We have the full range $0 \leq \theta \leq 2\pi$ for $\theta$ and $0 \leq \varphi \leq \pi/2$ for $\varphi$ in view of $z \geq 0$ and $0 \leq \rho \leq 2$ inside the sphere:

$$V = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.$$ 

(iii) $D$ is the solid bounded by the sphere $\rho = 1$ in the half-space $y \geq 0$;

Solution. Substituting $y = \rho \sin \varphi \sin \theta$ in $y \geq 0$ (or from the geometric considerations), we have $\sin \theta \geq 0$ and hence $0 \leq \theta \leq \pi$. Note that $0 \leq \varphi \leq \pi$ and hence $\sin \varphi$ is never negative. Finally $r$ varies between 0 and 1.

$$V = \int_0^{\pi} \int_0^\pi \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.$$ 

(iv) $D$ is the solid bounded below by the $xy$-plane, on the sides by the sphere $\rho = 1$, and above by the cone $\varphi = \pi/4$.

Solution. Bounded below by the $xy$-plane means $z \geq 0$, substituting $z = \rho \cos \varphi$, we have $0 \leq \varphi \leq \pi/2$ (note the maximal region for $\varphi$). Since $D$ is also bounded above by the cone $\varphi = \pi/4$, we get the limits $\pi/2 \leq \varphi \leq \pi/2$. There are no restrictions for $\theta$, meaning the full range $0 \leq \theta \leq 2\pi$.

$$V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^1 \rho^2 \sin \varphi \, d\rho \, d\varphi \, d\theta.$$