Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

S h e e t 11

Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Write $\iiint_D f(x, y, z)\, dV$ as iterated integral in cylindrical coordinates without evaluating (i.e. write it as iterated integral of a function in r, θ, z and set up the limits):

(i) $f(x, y, z) = 5$, D is the cylinder $x^2 + y^2 < 1$, $-1 \le z \le 1$;

Solution. Substituting the expressions $x = r\cos\theta$, $y = r\sin\theta$ for cylindrical coordinates, we have $0 \le r \le 1$ and no restrictions on θ, i.e. $0 \le \theta \le 2\pi$.

$$\iiint_D f(x, y, z)\, dV = \int_0^{2\pi} \int_0^1 \int_{-1}^1 5r\, dz\, dr\, d\theta.$$

(ii) $f(x, y, z) = x^2+y^2$, D is the circular cylinder whose base is the circle $(x-1)^2+y^2 = 1$ in the xy-plane and whose top lies in the plane $z = 2 + y$.

Solution. Substituting the expressions $x = r\cos\theta$, $y = r\sin\theta$ for cylindrical coordinates into the equation $(x - 1)^2 + y^2 = 1$, we obtain, after cancellation, $r = 2\cos\theta$. Thus, in the region D, we have $0 \le r \le 2\cos\theta$. This also put the restriction $\cos\theta \ge 0$ which means $-\pi/2 \le \theta \le \pi/2$. The additional equations $0 \le z \le 2 + y$ put no further restrictions on r and θ.

$$\iiint_D f(x, y, z)\, dV = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} \int_0^{2+r\sin\theta} r^2 \cdot r\, dz\, dr\, d\theta.$$

Exercise 2

Set up the iterated integral with correct limits that calculates the volume of the given solid D in spherical coordinates ρ, φ, θ without evaluating:

(i) D is the solid between the spheres $\rho = 1$ and $\rho = 2$;

Solution. We have the full ranges $0 \leq \theta \leq 2\pi$, $0 \leq \varphi \leq \pi$ for θ and φ and $1 \leq \rho \leq 2$ between the spheres.

$$V = \int_0^{2\pi} \int_0^{\pi} \int_1^{2} \rho^2 \ \mathsf{sin}\theta \, d\rho \, d\varphi \, d\theta.$$

(ii) D is the solid bounded by the sphere $\rho = 1$ in the half-space $z \geq 0$;

Solution. We have the full range $0 \leq \theta \leq 2\pi$ for θ and $0 \leq \varphi \leq \pi/2$ for φ in view of $z \geq 0$ and $0 \leq \rho \leq 2$ inside the sphere:

$$V = \int_0^{2\pi} \int_0^{\pi/2} \int_0^{1} \rho^2 \ \mathsf{sin}\varphi \, d\rho \, d\varphi \, d\theta.$$

(iii) D is the solid bounded by the sphere $\rho = 1$ in the half-space $y \geq 0$;

Solution. Substituting $y = \rho \, \mathsf{sin}\varphi \, \mathsf{sin}\theta$ in $y \geq 0$ (or from the geometric considerations), we have $\mathsf{sin}\theta \geq 0$ and hence $0 \leq \theta \leq \pi$. Note that $0 \leq \varphi \leq \pi$ and hence $\mathsf{sin}\varphi$ is never negative. Finally r varies between 0 and 1.

$$V = \int_0^{\pi} \int_0^{\pi} \int_0^{1} \rho^2 \mathsf{sin}\varphi \, d\rho \, d\varphi \, d\theta.$$

(iv) D is the solid bounded below by the xy-plane, on the sides by the sphere $\rho = 1$, and above by the cone $\varphi = \pi/4$.

Solution. Bounded below by the xy-plane means $z \geq 0$, substituting $z = \rho \mathsf{cos}\varphi$, we have $0 \leq \varphi \leq \pi/2$ (note the maximal region for φ). Since D is also bounded above by the cone $\varphi = \pi/4$, we get the limits $\pi/2 \leq \varphi \leq \pi/2$. There are no restrictions for θ, meaning the full range $0 \leq \theta \leq 2\pi$.

$$V = \int_0^{2\pi} \int_{\pi/4}^{\pi/2} \int_0^{1} \rho^2 \mathsf{sin}\varphi \, d\rho \, d\varphi \, d\theta.$$