Course 2E1 2004-05 (SF Engineers & MSISS & MEMS)

S h e e t 10

Due: in the tutorial sessions next Wednesday/Thursday

Exercise 1

Find the area in polar coordinates (r, θ) of the region R:

(i) R is the region inside the curve $r = \sqrt{1 - \sin\theta}$;

Solution. The function $\sqrt{1 - \sin\theta}$ is defined for all θ, so θ runs over its the full range: $0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{1 - \sin\theta}$.

$$A = \int_R dA = \int_R r\,dr\,d\theta = \int_0^{2\pi} \int_0^{\sqrt{1-\sin\theta}} r\,dr\,d\theta = \int_0^{2\pi} \frac{r^2}{2}\Big|_0^{\sqrt{1-\sin\theta}} d\theta$$

$$= \int_0^{2\pi} \frac{1 - \sin\theta}{2}\,d\theta.$$

The rest of the calculation here and later is standard and omitted.

(ii) R is the region inside the cardioid $r = 1 + \cos\theta$;

Solution.

$$A = \int_0^{2\pi} \int_0^{1+\cos\theta} r\,dr\,d\theta = \int_0^{2\pi} \frac{(1 + \cos\theta)^2}{2}\,d\theta.$$

(iii) R is the region common to the interior of the cardioids $r = 1+\cos\theta$ and $r = 1-\cos\theta$.

Solution. We have $1 + \cos\theta \ge 1 - \cos\theta$ when $\cos\theta \ge 0$, i.e. $-\pi/2 \le \theta \le \pi/2$. So in this range of θ, the coordinate r in the common region changes from 0 to the smaller number $1-\cos\theta$. The full region is symmetric with respect to the y-axes, hence the area is twice the are of the region $-\pi/2 \le \theta \le \pi/2$, $0 \le r \le 1 - \cos\theta$.

$$A = 2\int_{-\pi/2}^{\pi/2} \int_0^{1-\cos\theta} r\,dr\,d\theta = 2\int_{-\pi/2}^{\pi/2} \frac{(1 - \cos\theta)^2}{2}\,d\theta.$$

Exercise 2

Find the volume of the space region D:

(i) D is the pyramid bounded by the coordinate planes and the plane $x + 2y + z = 2$;

Solution. We integrate in the order $dz\,dy\,dx$. D is given by $x \geq 0$, $y \geq 0$, $z \geq 0$, $x+2y+z \leq 2$. The first integration variable is z, the others being fixed. Thus the limits are $0 \leq z \leq 2 - x - 2y$. Now we have to project to the xy-plane or, equivalently, to eliminate z. We have $x \geq 0$, $y \geq 0$ and $0 \leq 2 - x - 2y$. This gives the limits for the next variable y: $0 \leq y \leq 1 - x/2$. Finally, eliminating y, we obtain the range for x: $0 \leq x \leq 2$.

$$V = \int_D dV = \int_D dz\,dy\,dx = \int_0^2 \int_0^{1-x/2} \int_0^{2-x-2y} dz\,dy\,dx.$$

(ii) D is the prism bounded by the coordinate planes and the planes $x + y = 1$, $z = 1$;

Solution. We have $0 \leq z \leq 1$ for z, $0 \leq y \leq 1 - x$ for y and $0 \leq x \leq 1$ for x. Thus

$$V = \int_D dV = \int_0^1 \int_0^{1-x} \int_0^1 dz\,dy\,dx.$$

(iii) D is the region bounded the coordinate planes, the plane $y+z = 1$ and the cylinder $x = 1 - y^2$.

Solution. We have $0 \leq z \leq 1 - y$ for z. Eliminating z, we get the projection to xy-plane: $0 \leq y \leq 1$, $0 \leq x \leq 1 - y^2$. The equation $x = 1 - y^2$ is more covenient to solve for x so we can take x as the next variable, thus taking the order of integration $dz\,dx\,dy$. Hence we have the limits for x: $0 \leq x \leq 1 - y^2$. Finally, eliminating x, we obtain $0 \leq 1 - y^2$ or $-1 \leq y \leq 1$ and finally $0 \leq y \leq 1$ with the above inequality.

$$V = \int_D dV = \int_0^1 \int_0^{1-y^2} \int_0^{1-y} dz\,dx\,dy.$$