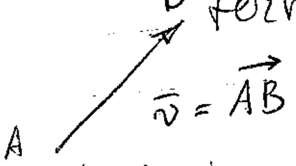
maths.tcd.ie/~zaitsev/... 2E02 2011 feedback! (1)

Linear Algebra

Book Anton & Rorres "Elementary Linear Algebra", Chap. 4-7

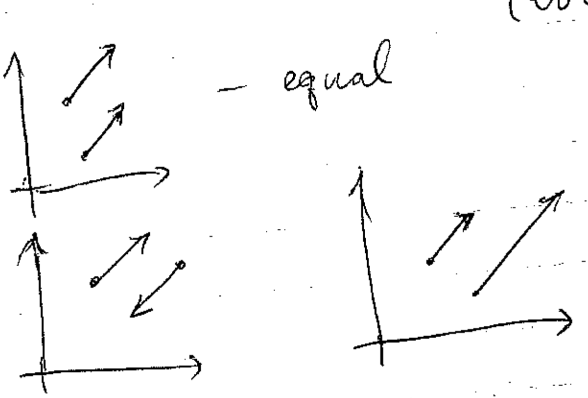
Review of vectors ~~and matrices~~ (Chap. 3)

Vectors = directed line segments or arrows (Ex. velocity, force, displacement)

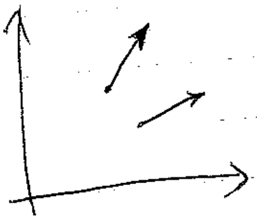


Vectors are equal if they have the same length and direction (even with different initials pts)

Length or norm $\|\bar{v}\| = |\vec{AB}|$



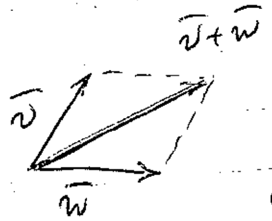
- equal



- not equal

Operations:

1) Sum



parallelogram rule

vector + vector = vector

2) Multiplication by a scalar:

$\bar{v}$ vector, k scalar $\Rightarrow$ $k\bar{v}$ is vector with legth $|k| \cdot \text{length}(\bar{v}) = |k| \cdot \|\bar{v}\|$,

direction of $k\bar{v}$ < the same as dir of $\bar{v}$ if k ; opposite to it if $k<0$

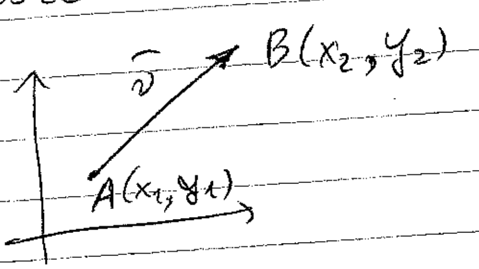
If $k=0$, then $k\bar{v} = 0$

scalar $\cdot$ vector = vector

Web, How to work, ... and Lect. Notes

Google Dmitri Zaitsev, then under Current Courses - Course 2E02

Coordinates in 2- and 3- spaces :

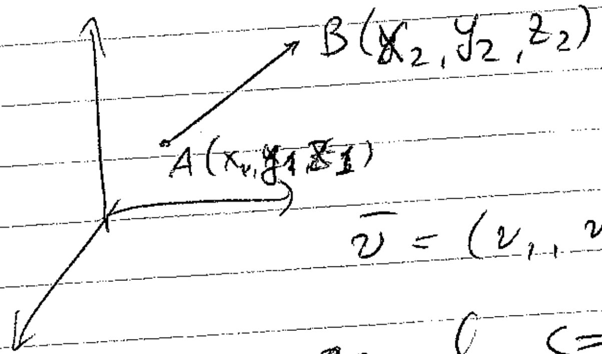


$\bar{v} = \overrightarrow{AB} = (x_2 - x_1, y_2 - y_1) = (v_1, v_2)$

components of $\bar{v}$

Vectors $\bar{v} = (v_1, v_2)$, $\bar{w} = (w_1, w_2)$ are equal $\Leftrightarrow$ the components are equal: $v_1 = w_1$, $v_2 = w_2$

Same in 3-space :



$\bar{v} = \overline{AB} = (x_2 - x_1, y_2 - y_1, z_2 - z_1)$
$= (v_1, v_2, v_3)$

$\bar{v} = (v_1, v_2, v_3)$ and $\bar{w} = (\ , \ , \ )$

are equal $\Leftrightarrow$ $v_1 = w_1$, $v_2 = w_2$, $v_3 = w_3$

<u>Length or norm</u> : $\bar{v} = (v_1, v_2) \Rightarrow \|\bar{v}\| = \sqrt{v_1^2 + v_2^2}$

$\bar{v} = (v_1, v_2, v_3) \Rightarrow \|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

Sum : $(v_1, v_2, v_3) + (w_1, w_2, w_3) = (v_1 + w_1, v_2 + w_2, \ldots$

Multiplication by k : $k(v_1, v_2, v_3) = (kv_1, kv_2, kv_3$

<u>Dot product of vectors</u>

$\bar{v} = (v_1, v_2)$, $\bar{w} = (w_1, w_2)$ $\quad \bar{v} \cdot \bar{w} = v_1 w_1 + v_2 w_2$

vector $\cdot$ vector $=$ scalar

Similar in 3-space : $(v_1, v_2, v_3) \cdot (w_1, w_2, w_3) = v_1 w_1 + \ldots$

If $\bar{v} \neq 0$, $\bar{w} \neq 0$:

$\bar{v} \cdot \bar{w} = \|\bar{v}\| \cdot \|\bar{w}\| \cdot \cos \theta$, where θ – angle between $\bar{v}$, $\bar{w}$

Chap 4.1) (Chips – input / output voltages, Mechanical Systems, Statistics ~~data~~

Vectors in n-space:

Ordered n-tuple is a sequence of n real numbers: $(a_1, a_2, \dots, a_n)$

$\mathbb{R}^n$ = Euclidean n-space = the sets of all ordered n-tuples

~~A~~ n-vector = vector in $\mathbb{R}^n$ = vector in n-spce = ordered n-tuple $\bar{v} = (v_1, v_2, \dots, v_n)$

$\bar{v} = (v_1, \dots, v_n)$, $\bar{w} = (w_1, \dots, w_n)$ are equal if $v_1 = w_1, \dots, v_n = w_n$

The sum: $\bar{v} + \bar{w} = (v_1 + w_1, \dots, v_n + w_n)$

Scalar multipl: $k\bar{v} = (kv_1, \dots, kv_n)$

zero vector: $\bar{0} = \underbrace{(0, \dots, 0)}_{n}$

Norm (or lenght) ~~or Euclidean norm~~:

$\|\bar{v}\| = \sqrt{v_1^2 + \dots + v_n^2}$ also calle Euclidean norm or length

Dot product (or Euclidean inner product):

$\bar{v} = (v_1, \dots, v_n)$, $\bar{w} = (w_1, \dots, w_n)$, $\bar{v} \cdot \bar{w} = v_1 w_1 + \dots$

~~$\bar{v} \cdot \bar{w} = \|\bar{v}\| \cdot \|\bar{w}\| \cos\theta$~~

The angle θ between $\bar{v}, \bar{w}$ can how be obtained fro ~~defined by~~

$\cos\theta = \dfrac{\bar{v} \cdot \bar{w}}{\|\bar{v}\| \cdot \|\bar{w}\|} = C$. That $|C| \leq 1$ is guaran

by Cauchy-Schwarz inequality: $|\bar{v} \cdot \bar{w}| \leq \|\bar{v}\| \cdot \|\bar{w}\|$

Ex. Evaluate $\bar{u} + \bar{v}$, $\|\bar{u} - 2\bar{v}\|$, $\bar{u} = (1, 0, 1, 0)$, $v = (-1, 0, 2, 1$

Consequence: $\|\bar{v}+\bar{w}\| \le \|\bar{v}\| + \|\bar{w}\|$

Indeed

Orthogonal vectors: $\bar{v} \neq 0$ and $\bar{w} \neq 0$ are orthog. if $\theta = \frac{\pi}{2}$
$\Leftrightarrow \cos\theta = 0 \Leftrightarrow \bar{v}\cdot\bar{w} = 0$

E.g. $(1,0,0,0)$, $(0,1,0,0)$ are orthog.
$(1,1,0,0)$ $(0,1,0,0)$ not orthog.

Ex. For which k are $\bar{u} = (k, 0, 1)$, $\bar{v} = (k, 5, 0)$ orth.

Review on linear systems and matrices

Lin syst: (Chap 1.1)
$$\begin{cases} 2x + y = 5 \\ 3z = 1 \\ x - z = 0 \end{cases} \;\equiv\; \begin{cases} 2x + y + 0z = 5 \\ 0x + 0y + 3z = 1 \\ x + 0y - z = 0 \end{cases}$$

Matrix notation:
$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} \text{ or } A\bar{u} = \bar{b}$$

Here $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & -1 \end{pmatrix}$ is 3×3 matrix

$\bar{u} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $\bar{b} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$ are vectors written as column

Matrix multiplication: row by column

$$\begin{array}{c} x \quad \text{row} \quad z \\ \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 3 \\ 1 & 0 & -1 \end{pmatrix} \end{array} \cdot \underset{\text{column}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} 2x + 1y + 0z \\ 0x + 0y + 3z \\ 1x + 0y - 1z \end{pmatrix}$$

General: (Chap 1.3)
$$A = \begin{pmatrix} a_{11} & \cdots & a_{1r} \\ a_{i1} & \cdots & a_{ir} \\ a_{m1} & & a_{mr} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & b_{1j} & b_{1n} \\ \vdots & \vdots & \\ b_{r1} & b_{rj} & b_{rn} \end{pmatrix}$$

$(m \times r \text{ matrix}) \quad (r \times n \text{ matrix}) = (m \times n) \text{ matrix}$

$$(AB)_{ij} = (a_{i1}, \dots, a_{ir}) \begin{pmatrix} b_{1j} \\ \vdots \\ b_{rj} \end{pmatrix} =$$

$$= a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{ir} b_{rj}$$

$$= \sum_{k=1}^{r} a_{ik} b_{kj}$$

Transpose matrix: $A^T = \begin{pmatrix} a_{11} & \dots & a_{m1} \\ \vdots & & \vdots \\ a_{1r} & \dots & a_{mr} \end{pmatrix}$

$\bar{u} = (u_1, \dots, u_n)$

2.1 $\bar{u}^T = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ (Chap. 4.1.)

Dot product: $\bar{v} = (v_1, \dots, v_n)$,

$$\underset{\text{dot product}}{\bar{u} \cdot \bar{v}} = \underset{\text{matrix prod.}}{\bar{v} \cdot \bar{u}^T} = \cancel{uv}\ (v_1, \dots, v_n) \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$$

Functions or maps or transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$: (Chap. 4.2)

$\bar{x} = (x_1, \dots, x_n) \in \mathbb{R}^n$, $\bar{w} = (w_1, \dots, w_m) \in \mathbb{R}^m$

$$\begin{cases} w_1 = f_1(x_1, \dots, x_n) \\ w_2 = f_2(\dots) \\ \dots \\ w_m = f_m(x_1, \dots, x_n) \end{cases}$$

Briefly: $\bar{w} = f(\bar{x})$

$f : \mathbb{R}^m \to \mathbb{R}^m$

Ex. $\begin{cases} w_1 = x_1 + x_2 \\ w_2 = x_2^2 \\ w_3 = e^{x_1} \end{cases}$ $\bar{w} = (w_1, w_2, w_3) = f(x_1, x_2)$

$f : \mathbb{R}^2 \to \mathbb{R}^3$

$$f(0,1) = (0+1, 1^2, e^1) = (1, 1, e)$$

Use of matrix version of dot product (5.1)

$$\cancel{\bar{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}} \qquad \bar{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\cancel{\bar{u} \cdot \bar{v} = \underbrace{(0\ 1\ 0) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (0\ 1\ 0) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2}$$

more convincing

$$\bar{u} = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 5 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \qquad v = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\underbrace{(0\ 1\ 0) \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 5 \end{pmatrix}} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = (2\ 1\ 3) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 2+2\cdot 3\cdot 3 =$$

Transformation $T_A \longleftrightarrow$ Matrix A (standard matrix of transf.

$$T(x_1, x_2) = (x_1, 2x_2, x_1 + x_2) = (w_1, w_2, w_3)$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_2 \\ x_1 + x_2 \end{pmatrix} = \begin{pmatrix} x_1 & 0x_2 \\ 0x_1 & 2x_2 \\ x_1 & + x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

A transformatio matrix of T

A transformation f is linear if all functions $f_1, \dots, f_m$ are linear:

$$\begin{cases} w_1 = a_{11}x_1 + \dots + a_{1n}x_n \\ \quad \dots \\ w_m = a_{m1}x_1 + \dots + a_{mn}x_n \end{cases} \quad \text{or} \quad \begin{pmatrix} w_1 \\ \vdots \\ w_m \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$\bar{w} = A\bar{x}$$

Examples f is given by the matrix A e.g.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \cdot x_1 + 0 \cdot x_2 \\ 0 \cdot x_1 + 2 \cdot x_2 \\ 1 \cdot x_1 + 1 \cdot x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_2 \\ x_1 + x_2 \end{pmatrix}$$

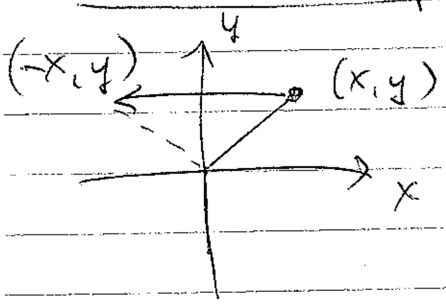
Ex. Calculate the value of f at $(x_1, x_2) = (1, -1)$

$$f(1,-1) = A \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \cdot 1 - 1 \cdot 0 \\ 1 \cdot 0 - 1 \cdot 2 \\ 1 \cdot 1 - 1 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ - \\ 0 \end{pmatrix}$$

Examples of linear transformations (Ch. 4.2) $T = T_A$

A standard matri of T

Reflection Operators

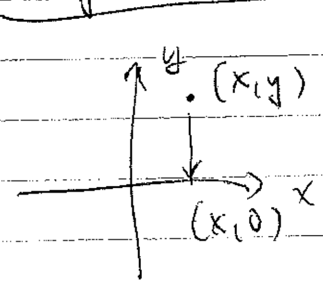


$f(x,y) = (-x, y) = (w_1, w_2)$ (x', y')

$$x' = -x = -1 \cdot x + 0y$$
$$y' = y = 0 \cdot x + 1 \cdot y \quad \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ matrix of the reflection about the y-a

Projection Operators $\rightarrow$ same in $\mathbb{R}^3$ onto planes

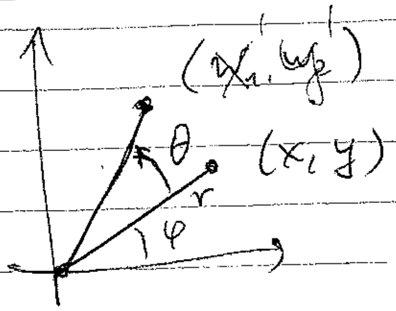


$f(x,y) = (x, 0) = (x', y')$

$$\begin{cases} x' = x = 1 \cdot x + 0 \cdot y \\ y' = 0 = 0 \cdot x + 0 \cdot y \end{cases} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Euler's formula e^{x+iy}

Rotation Operators;



$x = r\cos\varphi \qquad w_1' = r\cos(\theta+\varphi)$

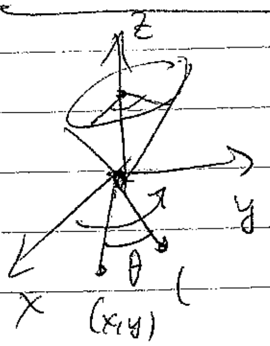
$y = r\sin\varphi \qquad w_2' = r\sin(\theta+\varphi)$

$= r\cos\theta\cos\varphi - r\sin\theta\sin\varphi$

$= r\sin\theta\cos\varphi + r\cos\theta\sin\varphi$

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Rotation in $\mathbb{R}^3$ About the z-axis:



$$\begin{cases} w_1' = x\cos\theta - y\sin\theta \\ w_2' = x\sin\theta + y\cos\theta \\ w_3' = z \end{cases} \qquad A = \begin{pmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ 0 & 0 & 1 \end{pmatrix}$$

$(x\,y\,z)$ $\quad x \to y \to z$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Dilation and Contraction

$T(\bar{x}) = k\bar{x}$ – contraction for $0 \le k \le 1$

– dilation for $k \ge 1$

$$\begin{cases} w_1' = kx = k\cdot x + 0y \\ w_2' = ky = 0x + ky \end{cases} \qquad A = \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$$

Composition $\quad T_A : \mathbb{R}^n \to \mathbb{R}^k, \; T_B : \mathbb{R}^k \to \mathbb{R}^m$

$$T_B \circ T_A(\bar{x}) = T_B(T_A(\bar{x})) = B(A\bar{x}) = (BA)\bar{x}$$

$T_B \circ T_A$ – linear transf. with matrix $B \cdot A$

Example: Composition of refl about x- and y-axes

Subspaces (Chap. 5.2)

8

$W = \mathbb{R}^n$ - the main space = ambient space

($\mathbb{R}^n$ = the set of n-tuples $(x_1, \dots, x_n)$)

Def. A subset V of W is called a subspace if it is closed under vector additions and multip. by scalars:

That is, if $\bar{u}, \bar{v} \in V$, then $\bar{u} + \bar{v} \in V$
and if $k \in \mathbb{R}$, $\bar{u} \in V$, then $k\bar{u} \in V$

~~Corollary If V is a subspace~~

Ex. (1) Lines in $\mathbb{R}^2$: ~~Given~~ Any $\bar{v} \neq 0$ in $\mathbb{R}^2$ defines the line $\ell_{\bar{v}} = \{t\bar{v} : t \in \mathbb{R}\}$ - the line in the direction of $\bar{v}$

ℓ is the subspace: $t_1\bar{v}, t_2\bar{v} \in \ell \Rightarrow \cancel{t_1 + t_2}$

$t_1\bar{v} + t_2\bar{v} = (t_1 + t_2)\bar{v} \in \ell$

$k \in \mathbb{R}$, $t\bar{v} \in \ell \Rightarrow k(t\bar{v}) = (kt)\bar{v} \in \ell$

~~(2) Similar~~ E.g. $\bar{v} = (1, 2)$, $\ell_{\bar{v}} = \{(1,2)t\} = \{(t, 2t\}$

$\ell_{(1,0)}$ is x-axis, $\ell_{(0,1)}$ is y-axis

~~(2) Similar~~

(2) Similar: Lines in $\mathbb{R}^3$

Ex. $\bar{v} = (1, 0, -3)$

(3) Planes in $\mathbb{R}^3$: one linear equation:

$x + 2y - 3z = 0$ is the plain V

$\bar{u}_1 = (x_1, y_1, z_1) \in V$,

Examples $\{(a, b, a+b)\}$, $\{(a, b, 1)\}$

$\{(x, y) : x > 0\}$ closed under sums but not under multiplication with -1

More general: Solutions Spaces of Homogeneous Linear Systems:

$$(*)\begin{cases} a_{11}x_1 + \dots + a_{1n}x_n = 0 \\ \dots \\ a_{m1}x_1 + \dots a_{mn}x_n = 0 \end{cases} \qquad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

If $\bar{x} = (x_1, \dots, x_n)$ is solution, $\bar{x}' = (x_1', \dots, x_n')$ another sol.
$\Rightarrow \bar{x} + \bar{x}' = (x_1 + x_1', \dots, x_n + x_n')$ is also a sol.
If $\bar{x}$ is sol. and $k \in \mathbb{R}$, then $k\bar{x} \dots$

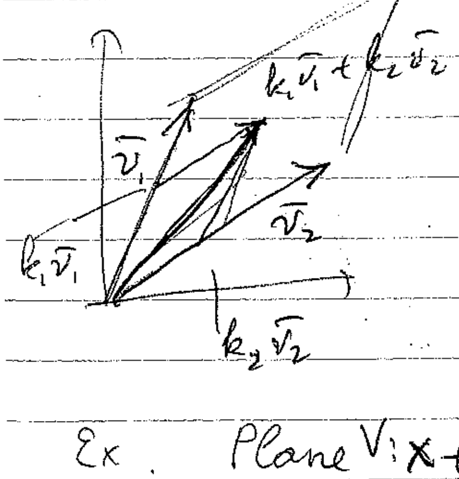
Short form: $(*) \Leftrightarrow A\bar{x} = 0$
So: $A\bar{x} = 0$, $A\bar{x}' = 0 \Rightarrow A(\bar{x} + \bar{x}') = A\bar{x} + A\bar{x}' = 0$
$A(\bar{x}) = 0$, $\Rightarrow A(k\bar{x}) = kA\bar{x} = 0$

Linear Combinations of vectors and their span

$\bar{v}_1, \bar{v}_2$ vectors, k_1, k_2 scalars
$\Rightarrow$ the vector $k_1\bar{v}_1 + k_2\bar{v}_2$ is called linear comb. of $\bar{v}_1, \bar{v}_2$

If V is a subspace, it contains all linear comb. of it vectors.

Ex: Plane V in $\mathbb{R}^3$, ~~Furthermore~~



More generally:

$k_1\bar{v}_1 + \dots + k_r\bar{v}_r$

Ex. Which are lin. comb. of $\bar{v}_1 = (0, -1, 1)$, $\bar{v}_2 = (1, 0, -1)$

(a) $(1, -1, 0)$
(b) $(1, 0, 0)$

Ex. Plane $V: x + y + z = 0$, solutions $\bar{v}_1 = (1, 0, -1)$
$v_2 = (0, 1, -1)$

Lin. comb. $k_1\bar{v}_1 + k_2\bar{v}_2 = \dots$
V = precisely all lin. comb.

Thm. Given any vectors $\bar{v}_1, \dots, \bar{v}_r$
the set of all lin comb. is a subspace
It is called or the span of $\bar{v}_1, \dots, \bar{v}_r$
the space spanned by ...
$V = \text{span}\{\bar{v}_1, \dots, \bar{v}_r\}$

Ex. $(0,1)$, $(1,0)$ span R^2
$(0,1)$, $(1,0)$, $(1,1)$ — " —
$(0,1)$, $(1,1)$ — " —
$(0,1)$, $(0,-1)$ span x-axis

~~(1,0,0)~~, In R^3 ...

Determine whether the vectors span R^3:
a) $\bar{v}_1 = (1,1,1)$, $\bar{v}_2 = (1,0,1)$, $\bar{v}_3 = (0,1,1)$

$(x,y,z) = k_1\bar{v}_1 + k_2\bar{v}_2 + k_3\bar{v}_3 = k(1,1,1) + k_2(\dots) + k_3$

$$\begin{cases} x = k_1 + k_2 \\ y = k_1 + k_3 \\ z = k_1 + k_2 + k_3 \end{cases} \quad \begin{aligned} k_2 &= z - y \\ k_3 &= z - x \\ k_1 &= x - k_2 = x - z + y \end{aligned} \quad \begin{cases} k_2 = x - k_1 \\ y = k_1 + k_3 \\ z = \cancel{k_1} + x - \cancel{k_1} + k_3 \end{cases}$$

Ex. (b) $(1,0,-1)$, $(0,1,0)$, $(1,1,-1)$

$$\begin{cases} x = k_1 + k_3 \\ y = k_2 + k_3 \\ z = -k_1 - k_3 \end{cases} \quad \begin{cases} k_1 = x - k_3 \\ y = k_2 + k_3 \\ z = -x + \cancel{k_3} - \cancel{k_3} \end{cases}$$

only solvable in (k_1, k_2, k_3) for $z = -x$.

Ex. (a) Find parametric eq. for the line spanned by $\bar{u} = (1,0,3)$
$l_{\bar{u}} = \{k\bar{u}\}$ ~~or~~ $\bar{x} = k\bar{u}$

(b) Find eq. for plane spanned by $\bar{v}_1 = (0,1,1)$, $\bar{v}_2 = (-1,0,1)$
~~$ax + by + cz = 0$~~ eliminate x, y, z

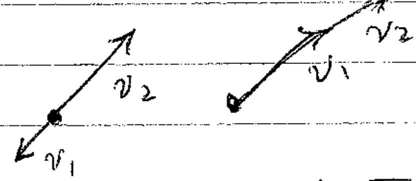
Linear Independence (Ch. 5.3)

$\{\bar{v}_1, \dots, \bar{v}_r\}$ is lin. indep. if the equation

$k_1\bar{v}_1 + \dots + k_r\bar{v}_r = 0$ has only the trivial solution $k_1 = \dots = k_r = 0$

Otherwise lin. dependent.

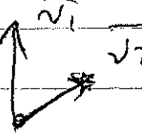
Ex.



$\bar{v}_2 = k\bar{v}_1 \Rightarrow k\bar{v}_1 - \bar{v}_2 = 0$

$\Rightarrow$ lin. dependent

Vice versa $k_1v_1 + k_2v_2 = 0$ ⋮



lin. indep.

$v_1(1,0,1)$, $v_2(0,1,0)$, $\bar{v}_3(1,1,1)$ are lin. dep.:

$v_1 + v_2 - v_3 = 0$

Ex. The unit ~~direction~~ vectors $\bar{i} = (1,0,0)$, $\bar{j} = (0,1,0)$, $\bar{k} = (0,0,1)$ are lin. indep.

Geom.

Thm. A set $\{\bar{v}_1, \dots, \bar{v}_r\}$ is lin dep. if and only if one is expressible as comb. of the others

Ex. ~~3 vector~~ A set of 3 vectors in R^2 is always lin. dep.

Thm. If $r > n$, any set of r vectors $\bar{v}_1, \dots, \bar{v}_r$ in is lin. dep.

Ex. Determine whether the vect. are lin. indep.:

(a) $\bar{v}_1 = (0,0,2)$, $\bar{v}_2 = (1,-1,0)$, $\bar{v}_3 = (2,-2,2)$

(b) $\bar{v}_1 = (0,0,2,0)$ $(1,-1,0,0)$ $(2,-2,2,1)$

$k_1\bar{v}_1 + k_2\bar{v}_2 + k_3 v_3 = 0$

(c) $\bar{v}_1 = (0,0,0)$, $\bar{v}_2 = (1,0,0)$: $\bar{v}_1 = 0 \Rightarrow$ always lin dep.

Geometric Interpret.:

In $\mathbb{R}^2$: do not lie in the same line

In $\mathbb{R}^3$ — " — plane

Ex. Determine whether the vect. are in the same plane

$\bar{v}_1 = (1,-1,0)$, $\bar{v}_2 = (-2,2,0)$, $\bar{v}_3 = (0,1,1)$

$2\bar{v}_1 - \bar{v}_2 = 0$

Same vector: (a) determine whether they are in the same li

(b) Are any two of them in the same line?

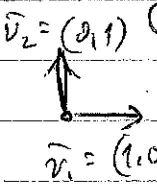
Basis and Dimension (Ch. 5.4)

lin. indep.

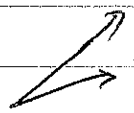
To span a line: need 1 vector
plane: two vectors not in one line
3-space: 3 vector not in one plane

<u>Def.</u> A basis in a subspace $V \subset R^m$ is a collection $\{\bar{v}_1, \dots, \bar{v}_n\}$ which
(a) spans V
(b) is linearly independent

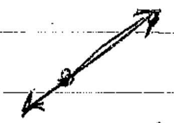
In R^2:



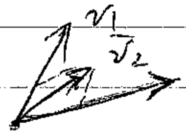
basis



basis



not basis:
does not span



not basis:
lin. dep.

<u>Basis Representation</u> If $S = \{\bar{v}_1, \dots, \bar{v}_n\}$ a basis,

every vector $\bar{v} \in V$ can be uniquely expressed as
$\bar{v} = c_1 \bar{v}_1 + \dots + c_n \bar{v}_n$

Can consider $(c_1, \dots, c_n)$ as "nonrectangular coordinates" relative to the basis S

Have one-to-one correspondence:

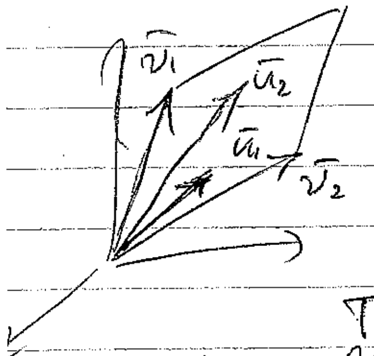
$$(c_1, \dots, c_n) \Leftrightarrow \bar{v} = c_1 \bar{v}_1 + \dots + c_n \bar{v}_n$$

(the order of coordinates depends on the order of $\bar{v}_i$)

<u>Ex.</u> Standard basis: $\bar{i} = (1,0)$, $\bar{j} = (0,1)$ in R^2
$\bar{i} = (1,0,0)$, $\bar{j}$, k in R^3
In R^n: $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, ...

~~Ex. Is $S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$ a basis?~~

Different bases in the same space:



Thm. All bases in the same space have the same number of vectors

This ~~numb unique~~ invariant number $\dim V \rightarrow$ is called the dimension of the space

$\dim \mathbb{R}^2 = 2$, $\dim \mathbb{R}^3 = 3$, $\dim \mathbb{R}^n = n$

Ex. Find dimension of the solution space:

$$\begin{cases} x + y - z = 0 \\ -2x + y = 0 \\ 3y - 2z = 0 \end{cases} \qquad \begin{cases} x = -y + z \\ 2y + 2z + y = 0 \\ 3y = 2z \end{cases}$$

$$\begin{cases} x = -y + z \\ y = -\frac{2}{3} z \\ 0 = 0 \end{cases} \qquad \begin{cases} x = \frac{5}{3} z \\ y = -\frac{2}{3} z \end{cases}$$

V - set of all solutions

z is free! choose $z = 1$

$$\begin{cases} x - y = 0 \\ y - 2z + t = 0 \end{cases} \qquad \begin{cases} x = y \\ y = 2z - t \end{cases} \qquad \begin{cases} x = 2z - t \\ y = 2z - t \end{cases}$$

(z, t) free! Choose $\bar{v}_1 = (x, y, z, t)$ with $z = 1, t = 0$

$x = 2$, $y = 2$ $\quad \bar{v}_1 = (2, 2, 1, 0)$

Choose $\bar{v}_2 = (x, y, z, t)$ with $z = 0, t = 1$

$x = -1 = y$ $\quad \bar{v}_2 = (-1, -1, 0, 1)$

<u>Useful thm.</u> If $\dim V = n$ and $S = \{\bar{v}_1, \dots, \bar{v}_n\}$,

then S basis if and only if S is lin. indep.
iff S spans V

Ex. Check if the following is a basis in $\mathbb{R}^2$:

(1) $\bar{v}_1 = (1, -1)$, $S = \{\bar{v}_1\}$
$\to$ only 1 vector $\to$ less than $\dim \mathbb{R}^2 = 2$
$\Rightarrow$ not a basis

(2) $S = \{\bar{v}_1, \bar{v}_2\}$, $\bar{v}_1 = (1, -1)$, $\bar{v}_2 = (-1, 2)$

Lin dependence: $k_1 \bar{v}_1 + k_2 \bar{v}_2 = 0$

$$\begin{cases} k_1 \cdot 1 - k_2 \cdot 1 = 0 \\ k_1(-1) + k_2 \cdot 2 = 0 \end{cases} \qquad \begin{cases} k_1 = k_2 \\ -k_1 + 2k_2 = 0 \end{cases}$$

$\Rightarrow k_2 = 0 \Rightarrow k_1 = 0 \Rightarrow$ only trivial solution $(k_1, k_2) = (0, 0)$

$\Rightarrow$ lin. indep. $\Rightarrow$ ~~not~~ a basis by thm. ↑

(3) $S = \{\bar{v}_1, \bar{v}_2, \bar{v}_3\}$, $\bar{v}_1 = (1, -1)$, $\bar{v}_2 = (-1, 2)$.

$\Rightarrow$ 3 vectors $\Rightarrow$ more than $\dim \mathbb{R}^2 = 2$
$\Rightarrow$ not a basis

Representing a Vector using a basis (Ch. 5.4) 16

Nonrectangular coordinates

Ex. Find the coordinates of $\bar{v}$ wrt. a basis

$\bar{v} = (1,2)$, basis $S = \{\bar{v}_1, \bar{v}_2\}$, $\bar{v}_1 = (0,1)$, $\bar{v}_2 = (1,-$

$\bar{v} = c_1 \bar{v}_1 + \bar{c}_2 \bar{v}_2$ $\begin{cases} 1 = c_1 \cdot 0 + c_2 \cdot 1 \\ 2 = c_1 \cdot 1 + c_2(-1) \end{cases}$ $\begin{cases} c_2 = 1 \\ 2 = c_1 - c_2 \end{cases}$

$c_1 = 2 + c_2 = 3$

The coordinates are $(1,3)$: $\bar{v} = 1 \cdot \bar{v}_1 + 3 \cdot \bar{v}_2$

~~Anothe~~ Same vector, anothe basis

$v_1 = (0,1)$, $v_2 = (1,1)$

$\begin{cases} 1 = c_1 \cdot 0 + c_2 \cdot 1 \\ 2 = c_1 \cdot 1 + c_2 \cdot 1 \end{cases}$ $\begin{cases} c_2 = 1 \\ 2 = c_1 + c_2 \end{cases}$ $c_1 = 1$

Coord : $(1,1)$: $\bar{v} = 1 \cdot \bar{v}_1 + 1 \cdot \bar{v}_2$

Gens

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Ex. $\begin{cases} x_1 - x_3 + x_4 = 1 \\ 2x_4 + x_2 = 0 \end{cases}$ $\begin{cases} x_1 = 1 + x_3 - x_4 \\ x_2 = -2x_4 \end{cases}$

$x_3 = t$, $x_4 = s$, $x_1 = 1 + t - s$, $x_2 = -2s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 1+t-s \\ -2s \\ t \\ s \end{pmatrix} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{= x_0} + t \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}}_{= \bar{v}_1} + s \underbrace{\begin{pmatrix} -1 \\ -2 \\ 0 \\ 1 \end{pmatrix}}_{= \bar{v}_2}$$

$\bar{x}$

particular solution

general sol.

Write as $A\bar{x} = \bar{b}$

$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix}$ $\Rightarrow$ $\{\bar{v}_1, \bar{v}_2\}$ is basis for the Nullspace of A

Ex. Find basis for the nullsp. of A:

$A = \begin{pmatrix} 0 & 1 & -1 & 2 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$ $\quad A \begin{pmatrix} x_1 \\ \vdots \\ x_5 \end{pmatrix} = 0$

$\cancel{x_1 \; x_2 \; x_3 \; x_4 \; x_5}$

$\begin{cases} x_2 - x_3 + 2x_4 = 0 \\ x_1 + x_2 = 0 \end{cases}$ $\begin{cases} x_1 = -x_2 \\ x_3 = x_2 + 2x_4 \end{cases}$

$x_2 = t$, $x_4 = s$, $x_5 = u$ $\Rightarrow$ $x_1 = -t$, $x_3 = t + 2s$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -t \\ \\ \\ \\ \end{pmatrix}$$