Exercise 1
Identify even or odd functions and find their Fourier series for the period $-1 \leq x \leq 1$:
(i) $f(x) = 2x$;
(ii) $f(x) = -x^2$.

Exercise 2
Use Fourier series to find a solution of the equation
\[ y''(x) - y(x) = a(x), \]
where $a(x) = 2x$ for $-1 \leq x \leq 1$.

Solution:
For the interval $[-1, 1]$ and the odd function $a(x)$ we calculate
\[ b_n = \int_{-1}^{1} a(x) \sin(n\pi x) \, dx = \int_{-1}^{1} 2x \sin(n\pi x) \, dx = \int_{-1}^{1} uu' \, dx \]
\[ = (uv)|_{-1}^{1} - \int_{-1}^{1} u'v \, dx = -2x \frac{\cos(n\pi x)}{n\pi} |_{-1}^{1} + 2 \int_{-1}^{1} \cos(n\pi x) \, dx \]
\[ = -\frac{4}{n\pi} \cos(n\pi) + 0 = -\frac{4(-1)^n}{n\pi}, \]
hence
\[ a(x) = \sum b_n \sin(n\pi x) = -\frac{4}{\pi} \sum \frac{(-1)^n}{n} \sin(n\pi x). \]

Next, for $y$ we use general Fourier Series
\[ y(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\pi x) + b_n \sin(n\pi x)), \]
and its second derivative
\[ y''(x) = - \sum_{n=1}^{\infty} \left( n^2 \pi^2 a_n \cos(n \pi x) + n^2 \pi^2 b_n \sin(n \pi x) \right), \]

and substitute in our differential equation. Identifying constant coefficients we obtain
\[ -a_0 = 0. \]

Identifying coefficients of \( \cos(n \pi x) \) we have
\[ n^2 \pi^2 a_n - a_n = 0, \]
whence
\[ a_n = 0. \]
Finally identifying coefficients of \( \sin(n \pi x) \) we obtain
\[ -(n^2 \pi^2 b_n - b_n) = - \frac{4(-1)^n}{n \pi}, \]
from where
\[ b_n = \frac{4(-1)^n}{n \pi(1 + n^2 \pi^2)} \]
and hence
\[ y(x) = \sum b_n \sin(n \pi x) = 4 \sum \frac{4(-1)^n}{n \pi(1 + n^2 \pi^2)} \sin(n \pi x). \]

**Exercise 3**

Find the Fourier integral representation of the function
\[ f(x) = \begin{cases} x & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1. \end{cases} \]

**Solution:**

Since the function is odd, we only need to compute the odd coefficient
\[ B(w) = \frac{1}{\pi} \int_{-1}^{1} x \sin wx \, dx = \frac{1}{\pi} \int_{-1}^{1} u' \, dv = \frac{1}{\pi} \left( \frac{-\cos wx}{w} \right)_{x=1}^{x=-1} + \int_{-1}^{1} \frac{\cos wx}{w} \, dx \]
\[ = \frac{1}{\pi} \left( \frac{-2 \cos w}{w} + \frac{\sin wx}{w^2} \right)_{x=1}^{x=-1} = \frac{1}{\pi} \left( \frac{-2 \cos w}{w} + \frac{2 \sin w}{w^2} \right), \]

and so the Fourier integral is
\[ f(x) = \int_{0}^{+\infty} B(w) \sin wx \, dw = \frac{2}{\pi} \int_{0}^{+\infty} \frac{\sin w - w \cos w}{w^2} \sin wx \, dw. \]