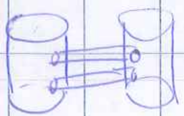
~~55~~ 37.1.

Applications of Linear Algebra to Diff. Equations (Ch. 8.1 Anton–Rorres)



Ex. $y' = ay$, $y = y(x)$ unknown function

$\Rightarrow$ solutions $y = ce^{ax}$ $\Rightarrow$ $y' = (ce^{ax})' = ay$

c – any constant! $\rightarrow$ General solution

Particular solution: $y(0) = 2 \Rightarrow ce^0 = 2 \Rightarrow c = 2$

y' = Inflow – Outflow

Ex. Systems
$$\begin{cases} y_1' = y_1 + y_2 \\ y_2' = 4y_1 - 2y_2 \end{cases} \qquad \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}}_{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$\rightarrow$ Solve ~~Eigenvalue – Eigenvector problem~~ Diagonalization Problem for A

① E-values: $\det(\lambda I - A) = \begin{vmatrix} \lambda - 1 & -1 \\ -4 & \lambda + 2 \end{vmatrix} = (\lambda + 3)(\lambda - 2)$

$\Rightarrow \lambda_1 = 2$, $\lambda_2 = -3$ E-values

② E-vectors: $\begin{pmatrix} \lambda - 1 & -1 \\ -4 & \lambda + 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\lambda = 2$ $\begin{pmatrix} 1 & -1 \\ -4 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\begin{cases} x_1 - x_2 = 0 \\ 0 = 0 \end{cases}$ $\Rightarrow$ Can take $x_2 = 1$ solution ~~$\Rightarrow x_1$~~

$\bar{e}_1 = (x_1, x_2) = (1, 1)$

$\lambda = -3$ $\begin{pmatrix} -4 & -1 \\ -4 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\Rightarrow \bar{e}_2 = (x_1, x_2) = (-1, 4)$

③ Diagonalization: $P^{-1}AP = D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix}$$
$\quad\;\; \vec{e}_1 \;\; \vec{e}_2$

④ Application to ODE system: $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = A\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\rightarrow \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = P^{-1}AP\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, ~~where~~ $\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = D\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = P\begin{pmatrix} u_1' \\ u_2' \end{pmatrix}$

$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \Leftrightarrow \begin{cases} u_1' = 2u_1 \\ u_2' = -3u_2 \end{cases} \Leftrightarrow \begin{cases} u_1 = c_1 e^{2x} \\ u_2 = c_2 e^{-3x} \end{cases}$

$\Leftrightarrow \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{2x} \\ c_2 e^{-3x} \end{pmatrix} \Rightarrow \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P\begin{pmatrix} c_1 e^{2x} \\ c_2 e^{-3x} \end{pmatrix} =$

$= \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} c_1 e^{2x} \\ c_2 e^{-3x} \end{pmatrix} = \begin{pmatrix} c_1 e^{2x} - c_2 e^{-3x} \\ c_1 e^{2x} + 4c_2 e^{-3x} \end{pmatrix}$

$\rightarrow$ General Solution

⑤ Particular Solution solving Initial val. Problem.

$\begin{cases} y_1(0) = 1 \\ y_2(0) = 6 \end{cases}$ $\quad$ $y_1(0) = c_1 e^0 - c_2 e^0 = c_1 - c_2 = 1$

$\quad$ $y_2(0) =$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\Big|_{x=0} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ $\quad$ $\begin{pmatrix} c_1 e^{2x} - c_2 e^{-3x} \\ c_1 e^{2x} + 4c_2 e^{-3x} \end{pmatrix}\Big|_{x=0} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$

$$\begin{pmatrix} c_1 - c_2 \\ c_1 + 4c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \iff \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \end{pmatrix} \Rightarrow \begin{cases} c_1 - c_2 = 1 \\ 3c_2 = 6 \end{cases}$$

$$\to c_2 = 2, \quad c_1 = 1 + c_2 = 3$$

$$\begin{cases} c_1 - c_2 = 1 \\ c_1 + 4c_2 = 6 \end{cases} \qquad \begin{cases} c_1 = c_2 + 1 \\ c_2 + 1 + 4c_2 = 6 \end{cases} \Rightarrow \begin{cases} c_1 = 2 \\ c_2 = 1 \end{cases}$$

$$\begin{cases} y_1 = 2e^{2x} - e^{-3x} \\ y_2 = 2e^{2x} + 4e^{-3x} \end{cases}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = e^{2x} \begin{pmatrix} 2 \\ 2 \end{pmatrix} \cancel{e^{2x}} + e^{-3x} \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

General $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = c_1 e^{2x} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-3x} \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

Inhomogneous $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} e^{x} \\ e^{-x} \end{pmatrix}$