Exercises 1
Determine which of the following are subspaces:

(i) the set of all vectors of the form \((a, 2a)\).

(ii) the set of all vectors of the form \((a, 0, a + b)\).

(iii) the set of all vectors of the form \((b, 2a, -b, 1 - b)\).

Subspaces:
\(S\) is a subspace if both of the following conditions are met:

(i) If \(v_1 \in S\) and \(v_2 \in S\), then \(v_1 + v_2 \in S\)

(ii) If \(v \in S\) and \(k \in \mathbb{R}\), then \(kv \in S\).

(i) \((a, 2a)\):
Test for vector addition:

\[
v_1 = \begin{pmatrix} a \\ 2a \end{pmatrix} \in S
v_2 = \begin{pmatrix} b \\ 2b \end{pmatrix} \in S
v_1 + v_2 = \begin{pmatrix} a + b \\ 2(a + b) \end{pmatrix} = \begin{pmatrix} c \\ 2c \end{pmatrix}
v_1 + v_2 \in S
\]

Test for scalar multiplication:

\[
v = \begin{pmatrix} a \\ 2a \end{pmatrix} \in S
kv = \begin{pmatrix} k(a) \\ k(2a) \end{pmatrix} = \begin{pmatrix} ka \\ 2(ka) \end{pmatrix} = \begin{pmatrix} d \\ 2d \end{pmatrix}
kv \in S
\]

Both conditions are met, therefore it is a subspace.
(ii) \((a, 0, a + b)\):
Test for vector addition:

\[
\begin{align*}
v_1 &= \begin{pmatrix} a \\ 0 \\ a + b \end{pmatrix} \in S \\
v_2 &= \begin{pmatrix} c \\ 0 \\ c + d \end{pmatrix} \in S \\
v_1 + v_2 &= \begin{pmatrix} (a + c) \\ 0 \\ (a + c) + (b + d) \end{pmatrix} = \begin{pmatrix} f \\ 0 \\ f + g \end{pmatrix} \\
v_1 + v_2 &\in S
\end{align*}
\]

Test for scalar multiplication:

\[
\begin{align*}
v &= \begin{pmatrix} a \\ 0 \\ a + b \end{pmatrix} \in S \\
kv &= \begin{pmatrix} ka \\ 0 \\ ka + kb \end{pmatrix} = \begin{pmatrix} (ka) \\ 0 \\ (ka) + (kb) \end{pmatrix} = \begin{pmatrix} h \\ 0 \\ h + k \end{pmatrix} \\
kv &\in S
\end{align*}
\]

Both conditions are met, therefore \textbf{it is a subspace}.

(iii) \((b, 2a, -b, 1 - b)\):
Test for scalar multiplication:

\[
\begin{align*}
v &= \begin{pmatrix} b \\ 2a \\ -b \\ 1 - b \end{pmatrix} \in S \\
kv &= \begin{pmatrix} kb \\ 2ka \\ -kb \\ k - kb \end{pmatrix} \\
c &= ka \\
d &= kb \\
kv &= \begin{pmatrix} d \\ 2c \\ -d \\ k - d \end{pmatrix}
\end{align*}
\]

\(kv \in S\) only when \(k = 1\).

Therefore \textbf{it is not a subspace}.
Exercises 2
Which of the following sets of vectors are linearly dependent?

(i) (1, −1), (−1, 0)
(ii) (0, 1, 1), (1, −1, 0), (−2, 0, −2)
(iii) (−1, 0, 1, 0, 0), (0, 2, 3, 1, 1), (0, −2, 0, 0, 1)

Linear independence:
A set of vectors \( \{v_1, \ldots, v_i, \ldots, v_k\} \) is linearly independent if:
\[
t_1 v_1 + \ldots + t_i v_i + \ldots + t_k v_k = 0
\]
only when \( t_i = 0 \) for all \( i \).
In each of these exercises we will determine linear independence by showing whether the above equation can be satisfied by only the trivial solution \( t_i = 0 \) for all \( i \) or otherwise.

(i) \((1, −1), (−1, 0)\):

\[
t_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

\[
\begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Using Gaussian elimination:

\[
\begin{pmatrix} 1 & 0 & : & 0 \\ 0 & 1 & : & 0 \end{pmatrix}
\]

\[
t_1 = 0
\]
\[
t_2 = 0
\]

The only solution is the trivial solution.
Therefore this set of vectors is linearly independent, it is not linearly dependent.

(ii) \((0, 1, 1), (1, −1, 0), (−2, 0, −2)\):

\[
t_1 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + t_3 \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]

\[
\begin{pmatrix} 0 & 1 & -2 \\ 1 & -1 & 0 \\ 1 & 0 & -2 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\]
Using Gaussian elimination:

\[
\begin{pmatrix}
1 & 0 & -2 : 0 \\
0 & 1 & -2 : 0 \\
0 & 0 & 0 : 0
\end{pmatrix}
\]

\[t_1 = 2t_3\]

\[t_2 = 2t_3\]

\(t_3\) is a free variable and allows for solutions other than the trivial one. Therefore this set of vectors is not linearly independent, it is \textcolor{red}{linearly dependent}.

(iii) \((-1, 0, 1, 0, 0), (0, 2, 3, 1, 1), (0, -2, 0, 1)\):

\[
\begin{pmatrix}
-1 \\
0 \\
1 \\
0
\end{pmatrix} + t_2 \begin{pmatrix}
0 \\
2 \\
1 \\
1
\end{pmatrix} + t_3 \begin{pmatrix}
0 \\
-2 \\
0 \\
1
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

Using Gaussian elimination:

\[
\begin{pmatrix}
1 & 0 & 0 : 0 \\
0 & 1 & 0 : 0 \\
0 & 0 & 1 : 0 \\
0 & 0 & 0 : 0
\end{pmatrix}
\]

\[t_1 = 0\]

\[t_2 = 0\]

\[t_3 = 0\]

The only solution is the trivial solution. Therefore this set of vectors is linearly independent, it is \textcolor{red}{not linearly dependent}.
Exercises 3
Which of the following sets of vectors are bases for the corresponding space \( \mathbb{R}^n \)?

(i) \((1, 1)\)
(ii) \( (1, 0), (1, -2) \)
(iii) \((1, 1), (2, 2)\)
(iv) \((1, -1), (15, 222), (-1, 1)\)
(v) \((1, -1, 2, 0), (1, 1, 5, -3), (1, -1, 2, 1)\)
(vi) \((1, 0, 1), (1, 1, 0), (2, 1, 0)\)

Basis:
A set of vectors \( S = \{v_1, \ldots, v_k\} \) is a basis for a space \( V \) if:

(i) \( \{v_1, \ldots, v_k\} \) spans the space.
(ii) \( \{v_1, \ldots, v_k\} \) are linearly independent.

For the space \( \mathbb{R}^n \), we need a set of \( n \) \( n \)-dimensional vectors to form a basis.
If \( k < n \) where \( k \) refers to the number of vectors in the set, the vectors will not span the space.
If \( k > n \), the vectors will not be linearly independent.

In this exercise if we find \( k \neq n \), we can immediately conclude this set of vectors does not form a basis for \( \mathbb{R}^n \).
If \( k = n \), we will check if the set is linearly independent. If so, we conclude the set does form a basis for \( \mathbb{R}^n \).

(i) \((1, 1)\):
\[
k = 1 \quad n = 2 \quad k < n
\]

\( k \neq n \) and thus this vector is **not a basis for \( \mathbb{R}^2 \)**.

(ii) \((1, 0), (1, -2)\):
\[
k = n = 2
\]

We will now check for linear independence i.e check if the trivial solution is the only solution to the following equation:

\[
t_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]
Using Gaussian elimination:

\[
\begin{pmatrix}
1 & 0 & : & 0 \\
0 & 1 & : & 0
\end{pmatrix}
\]

\[
t_1 = 0 \\
t_2 = 0
\]

The only solution is the trivial solution. Therefore this set of vectors is linearly independent. Therefore these vectors form a basis for \( \mathbb{R}^2 \).

(iii) \((1, 1), (2, 2)\):

\[
k = n = 2
\]

We will now check for linear independence i.e check if the trivial solution is the only solution to the following equation:

\[
t_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

Using Gaussian elimination:

\[
\begin{pmatrix}
1 & 2 & : & 0 \\
0 & 0 & : & 0
\end{pmatrix}
\]

\[
t_1 = -2t_2
\]

\( t_2 \) is a free variable and allows for solutions other than the trivial one. Therefore this set of vectors is not linearly independent. Therefore these vectors do not form a basis for \( \mathbb{R}^2 \).

(iv) \((1, -1), (15, 222), (-1, 1)\):

\[
k = 3 \\
n = 2 \\
k > n
\]

\( k \neq n \) and thus these vectors do not form a basis for \( \mathbb{R}^3 \).
(v) \((1, -1, 2, 0), (1, 1, 5, -3), (1, -1, 2, 1)\):

\[ k = 3 \]
\[ n = 4 \]
\[ k < n \]

\(k \neq n\) and thus these vectors do **not form a basis for** \(\mathbb{R}^4\).

(vi) \((1, 0, 1), (1, 1, 0), (2, 1, 0)\):

\[ k = n = 3 \]

We will now check for linear independence i.e check if the trivial solution is the only solution to the following equation:

\[
\begin{bmatrix}
1 & 0 & 2 \\
0 & 1 & 1 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
t_1 \\
t_2 \\
t_3
\end{bmatrix}
=
\begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix}
\]

Using Gaussian elimination:

\[
\begin{bmatrix}
1 & 0 & 1 & : & 0 \\
0 & 1 & 0 & : & 0 \\
0 & 0 & 1 & : & 0
\end{bmatrix}
\]

\[
t_1 = 0 \\
t_2 = 0 \\
t_3 = 0
\]

The only solution is the trivial solution. Therefore this set of vectors is linearly independent. Therefore these vectors **form a basis for** \(\mathbb{R}^3\).