# MA2E01 2016
# Tutorial 8 solutions

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**Disclaimer**: Some minor mistakes may be present in these solutions. Please email any corrections you find to croninro@tcd.ie

## Exercise 1

### Question

Find the eigenvalues and corresponding eigenvectors of the following matrix:

$$A = \begin{pmatrix} -1 & 2 & -1 \\ 0 & 2 & -3 \\ 0 & -4 & 6 \end{pmatrix}$$

### Solution

#### Eigenvalues

The eigenvalues are the $\lambda$'s such that $det(\lambda I - A) = 0$

So we need to solve $p(\lambda) = det(\lambda I - A) = 0$

$$p(\lambda) \;=\; det \begin{pmatrix} \lambda + 1 & -2 & 1 \\ 0 & \lambda - 2 & 3 \\ 0 & 4 & \lambda - 6 \end{pmatrix}$$

$$
\begin{aligned}
&= (\lambda+1)[(\lambda-2)(\lambda-6)-4(3)] \\
&= (\lambda+1)[\lambda^2-8\lambda+12-12] \\
&= (\lambda+1)(\lambda)(\lambda-8)
\end{aligned}
$$

And so $\lambda = -1,\ 0,\ 8$

**Eigenvectors**

Now that we have the eigenvalues, we need to find the eigenvectors $\vec{x}$ such that $\vec{x} \neq 0$ and $A\vec{x} = \lambda\vec{x}$.

This is equivalent to solving $\lambda\vec{x} - A\vec{x} = (\lambda I - A)\vec{x} = 0$

which is equivalent to finding a vector $\vec{x}$ in the null space of $\lambda I - A$

Since a matrix and its row reduced form are row equivalent, this means the null space of a matrix and the null space of its row reduced form are equal.

Therefore we can row reduce $\lambda I - A$ before solving to make it easier (if we like)

$\lambda = -1$

$$
\begin{aligned}
-I - A &= \begin{pmatrix} 0 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 4 & -7 \end{pmatrix} \\
&\to \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 4 & -7 \end{pmatrix} \\
&\to \begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

So to solve for an eigenvector we can find one vector in the nullspace of this matrix:

$$\begin{pmatrix} 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 - \frac{v_3}{2} = 0 \Rightarrow v_2 = 0$$

$$v_3 = 0$$

There is no condition imposed on $v_1$ and so it is a free variable. We only need one vector from the null space to be our eigenvector so we can arbitrarily assign it a value (with the only condition being $\vec{x} \neq 0$).

Here we choose 1.

This gives the first eigenvector as:

$$\vec{x_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$\lambda = 0$

$$\begin{aligned} -A &= \begin{pmatrix} 1 & -2 & 1 \\ 0 & -2 & 3 \\ 0 & 4 & -6 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

So we solve for a vector in the nullspace:

$$\begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_1 - 2v_3 = 0 \Rightarrow v_1 = 2v_3$$
$$v_2 - \frac{3v_3}{2} = 0 \Rightarrow v_2 = \frac{3v_3}{2}$$

Here we see that both $v_1$ and $v_2$ can be expressed in terms of $v_3$, therefore these two equations which represent our eigenvector will hold for all $v_3$.

So $v_3$ is our free variable, which we can arbitrarily choose (again apart from $v_3 = 0$) to give us an eigenvector.

Since the second equation has 2 in the denominator we choose $v_3 = 2$ for convenience.

This gives our second eigenvector:

$$\vec{x_2} = \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix}$$

$\lambda = 8$

$$\begin{aligned} 8I - A &= \begin{pmatrix} 9 & -2 & 1 \\ 0 & 6 & 3 \\ 0 & 4 & 2 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & -\frac{2}{9} & \frac{1}{9} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \\ &\rightarrow \begin{pmatrix} 1 & 0 & \frac{2}{9} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

So we solve for a vector in the nullspace:

$$\begin{pmatrix} 1 & 0 & \frac{2}{9} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_1 - \frac{2v_3}{9} = 0 \Rightarrow v_1 = -\frac{2v_3}{9}$$

$$v_2 + \frac{v_3}{2} = 0 \Rightarrow v_2 = -\frac{v_3}{2}$$

Again $v_3$ is our free variable, we choose 18 this time as it is the common denominator.

This gives our third eigenvector:

$$\vec{x_3} = \begin{pmatrix} -4 \\ -9 \\ 18 \end{pmatrix}$$

## Exercise 2

### Question

Find a matrix $P$ and a diagonal matrix $D$ diagonalizing A, i.e. $P^{-1}AP = D$, where $A$ is as in Exercise 1.

### Answer

This matrix $P$ is simply a matrix with our three eigenvectors as the columns:

$$P = \begin{pmatrix} 1 & 4 & -4 \\ 0 & 3 & -9 \\ 0 & 2 & 18 \end{pmatrix}$$

Then the diagonal matrix is one with our eigenvalues on the diagonal and zeros elsewhere.

**Caution**: be sure that that the eigenvalue on the first diagonal spot corresponds to the eigenvector in the first column of $P$, the second eigenvalue corresponds to the eigenvector in the second column of $P$ and similarly for the third.

$$D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

Now these matrices will satisfy $P^{-1}AP = D$

**Note**: if you wish to check your answer it may be easier to check that $AP = PD$, rather than calculating $P^{-1}$

# Exercise 3

## Question

Use Exercise 2 to solve the system of ordinary differential equations

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} = A \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

where A is as in Exercise 1.

## Answer

We consider the system:

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = D \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

Then substituting $D$ for $P^{-1}AP$ we get:

$$\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = P^{-1}AP \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$P\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = AP\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

We can see that if we consider $\vec{u}$ to be such that $P\vec{u} = \vec{y}$ and hence $P\vec{u'} = \vec{y'}$, that we get our original system

Therefore if we solve for $\vec{u}$, we can find $\vec{y} = P\vec{u}$.

So

$$\begin{aligned}
\vec{u'} &= D\vec{u} \\
\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\
\begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} &= \begin{pmatrix} -u_1 \\ 0 \\ 8u_3 \end{pmatrix}
\end{aligned}$$

And thus we can solve these simple ODEs to get:

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} c_1e^{-t} \\ c_2 \\ c_3e^{8t} \end{pmatrix}$$

where the $c_i$'s are arbitrary constants for which any value of them will solve the system.

**Note**: the $t$ is the independent variable for which we assume y depends on i.e. $y = y(t)$ and likewise $u = u(t)$.

Now we have $\vec{u}$, we simply find $\vec{y}$ as:

$$\begin{aligned}
\vec{y} &= P\vec{u} \\
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 1 & 4 & -4 \\ 0 & 3 & -9 \\ 0 & 2 & 18 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \\
\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &= \begin{pmatrix} 1 & 4 & -4 \\ 0 & 3 & -9 \\ 0 & 2 & 18 \end{pmatrix}\begin{pmatrix} c_1e^{-t} \\ c_2 \\ c_3e^{8t} \end{pmatrix}
\end{aligned}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} - 4c_2 - 4c_3 e^{8t} \\ 3c_2 - 9c_3 e^{8t} \\ 2c_2 + 18c_3 e^{8t} \end{pmatrix}$$