Course 2E01 2015 (SF Engineers & MSISS & MEMS)

S h e e t 9

Due: at the end of the tutorial

Exercise 1

Identify even or odd functions and find their Fourier series for the period $-1 \leq x \leq 1$:

(i) $f(x) = 2x$;

(ii) $f(x) = -x^2$.

Exercise 2

Use Fourier series to find a solution of the equation

$$y''(x) - y(x) = a(x),$$

where $a(x) = 2x$ for $-1 \leq x \leq 1$.

Solution:

For the interval $[-1, 1]$ and the odd function $a(x)$ we calculate

$$b_n = \int_{-1}^{1} a(x) \sin(n\pi x)\, dx = \int_{-1}^{1} 2x \sin(n\pi x)\, dx = \int_{-1}^{1} uv'\, dx$$

$$= (uv)|_{-1}^{1} - \int_{-1}^{1} u'v\, dx = -2x \frac{\cos(n\pi x)}{n\pi}|_{-1}^{1} + 2\int_{-1}^{1} \cos(n\pi x)\, dx$$

$$= -\frac{4}{n\pi}\cos(n\pi) + 0 = -\frac{4(-1)^n}{n\pi},$$

hence

$$a(x) = \sum b_n \sin(n\pi x) = -\frac{4}{\pi} \sum \frac{(-1)^n}{n} \sin(n\pi x).$$

Next, for y we use general Fourier Series

$$y(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(n\pi x) + b_n \sin(n\pi x)),$$

and its second derivative

$$y''(x) = -\sum_{n=1}^{\infty}(n^2\pi^2 a_n \mathsf{cos}(n\pi x) + n^2\pi^2 b_n \mathsf{sin}(n\pi x)),$$

and substitute in our differential equation. Identifying constant coefficients we obtain

$$-a_0 = 0.$$

Identifying coefficients of $\mathsf{cos}(n\pi x)$ we have

$$n^2\pi^2 a_n - a_n = 0,$$

whence

$$a_n = 0.$$

Finally identifying coefficients of $\mathsf{sin}(n\pi x)$ we obtain

$$-(n^2\pi^2 b_n - b_n) = -\frac{4(-1)^n}{n\pi},$$

from where

$$b_n = \frac{4(-1)^n}{n\pi(1+n^2\pi^2)}$$

and hence

$$y(x) = \sum b_n \mathsf{sin}(n\pi x) = 4\sum \frac{4(-1)^n}{n\pi(1+n^2\pi^2)}\mathsf{sin}(n\pi x).$$

Exercise 3

Find the Fourier integral representation of the function

$$f(x) = \begin{cases} x \text{ if } |x| < 1 \\ 0 \text{ if } |x| > 1. \end{cases}$$

Solution:

Since the function is odd, we only need to compute the odd coefficient

$$B(w) = \frac{1}{\pi}\int_{-1}^{1} x\,\mathsf{sin}wx\,dx = \frac{1}{\pi}\int_{-1}^{1} uv'\,dx$$

$$= \frac{1}{\pi}\left((uv)|_{x=-1}^{x=1} - \int_{-1}^{1} u'v\,dx\right) = \frac{1}{\pi}\left((x\frac{-\mathsf{cos}wx}{w})|_{x=-1}^{x=1} + \int_{-1}^{1}\frac{\mathsf{cos}wx}{w}\,dx\right)$$

$$= \frac{1}{\pi}\left(\frac{-2\mathsf{cos}w}{w} + \frac{\mathsf{sin}wx}{w^2}|_{x=-1}^{x=1}\,dx\right) = \frac{1}{\pi}\left(\frac{-2\mathsf{cos}w}{w} + \frac{2\mathsf{sin}w}{w^2}\right),$$

and so the Fourier integral is

$$f(x) = \int_0^{+\infty} B(w)\mathsf{sin}wx\,dw = \frac{2}{\pi}\int_0^{+\infty}\frac{\mathsf{sin}w - w\mathsf{cos}w}{w^2}\mathsf{sin}wx\,dw.$$