General and Particular Solutions (Ch. 5.5) 18

$A\bar{x} = \bar{b}$ $\rightsquigarrow$ $\bar{x}_0$ particular solution = any sol.
Any other solution $\bar{x}$ can be written as

$\bar{x} = \bar{x}_0 + \bar{v}$ with $A\bar{v} = 0$, i.e.
$\bar{v}$ is any solution of the associated homog. system.

$A\bar{x}_0 = \bar{b}$, $A\bar{v} = 0$ $\Rightarrow$ $A\bar{x} = A(x_0 + \bar{v}) = \bar{b}$

The solution space $\{\bar{v} : A\bar{v} = 0\}$ is a subspace called the Nullspace of A
Fix a Basis $\{\bar{v}_1, \dots, v_k\}$ of NA.

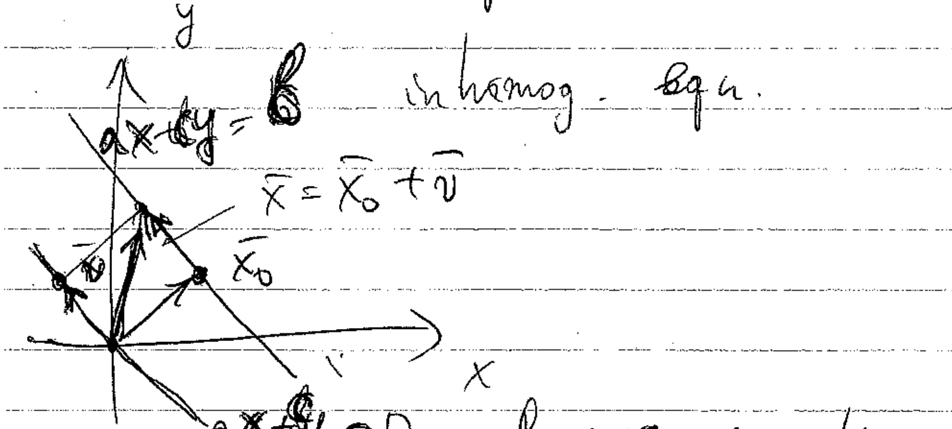
~~Any~~ General solution:

Of $A\bar{x} = 0$: $\bar{x} = c_1\bar{v}_1 + \dots + c_k\bar{v}_k$

Of $A\bar{x} = b$: $\bar{x} = x_0 + c_1\bar{v}_1 + \dots + c_k\bar{v}_k$,

where x_0 is a particular sol.

Geometric interpretation.



Row, Column and Nullspace (Ch. 5.5)

Matrix
$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

Rows of A: $\bar{r}_1 = (a_{11}, \dots, a_{1n}) \in \mathbb{R}^n$

$- - -$

$\bar{r}_m = (a_{m1} \dots a_{mn})$

Columns of A: $\bar{c}_1 = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix}, \dots, c_n = \dots \in \mathbb{R}^m$

Ex. $A = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 0 \end{pmatrix}$ $\bar{r}_1 = \dots$

Def. Row space of A = subspace spanned by rows (in $\mathbb{R}^n$)
Column —"— —"— columns
Nullspace —"— = solution space of $A\bar{x} = \bar{0}$

When is $A\bar{x} = \bar{b}$ solvable? If $\exists$ sol. $\bar{x}$,

$A\bar{x} = x_1\bar{c}_1 + \dots + x_n\bar{c}_n = \bar{b} \Rightarrow \bar{b}$ is lin. comb.

$\Rightarrow \bar{b} \in \operatorname{span}\{\bar{c}_1, \dots, \bar{c}_n\}$ = Column space

Thm. A linear system $A\bar{x} = \bar{b}$ ~~is solvable~~ has a solution (is consistent) if and only if $\bar{b}$ is in the column space of A

More efficient for big systems:

elementary row operations:

$$\begin{cases} 2x_2 - x_4 = 0 \\ x_1 + x_2 + x_3 = 0 \\ 2x_1 + 2x_2 + 2x_3 + x_4 = 0 \end{cases} \to \text{matrix} \quad A = \begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{pmatrix}$$

1) exchange the rows $\bar{r}_1, \bar{r}_2$:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 2 & 2 & 2 & 1 \end{pmatrix}$$

2) subtract ~~from~~ $2\bar{r}_1$ from $\bar{r}_3$:

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \text{~~row echelon form~~}$$

3) ~~replace~~ divide $\bar{r}_2$ by 2:

$$\hat{A} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to \text{row echelon form}$$

A and $\hat{A}$ are called row equivalent

- Row eq. matrices have the same Nullspace
- — " — Row space

(Note: Column Spaces may not be the same!

~~But relations for etc~~

But we can still compare bases! $\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} \boxed{1} & 1 \\ 0 & 0 \end{pmatrix}$

Finding bases for Row and Column Spaces

<u>Step 1.</u> Use elementary row operation to bring the matrix A to row echelon form $\hat{A}$

<u>Step. 2.</u> (a) The row vectors with the leading 1's form a basis for the row space of $\hat{A}$ and hence for that of A

(b) The column vectors ~~with the leading 1's~~ containing the above leading 1's form a basis for the column space of $\hat{A}$ (~~not~~ maybe not of A)

But: The column vectors of A having the same positions will form a basis for the column space of A

<u>Ex.</u> Find bases for row and col. space for A

$$A = \begin{array}{c} \begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \end{pmatrix} \\ \begin{array}{cccc} c_1 & c_2 & & c_4 \end{array} \end{array} \longleftrightarrow \tilde{A} = \begin{array}{c} \begin{array}{cc} \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 1 \end{pmatrix} & \begin{array}{c} \tilde{r}_1 \\ \tilde{r}_2 \\ \tilde{r}_3 \end{array} \end{array} \\ \begin{array}{cccc} \tilde{c}_1 & \tilde{c}_2 & & \tilde{c}_4 \end{array} \end{array}$$

$\{\tilde{r}_1, \tilde{r}_2, \tilde{r}_3\}$ basis for the R. Sp. of $\tilde{A}$ and the R. Sp. of A

Same positions!

$\{\tilde{c}_1, \tilde{c}_2, \tilde{c}_4\}$ basis for Col-Sp. of $\tilde{A}$

$\{c_1, c_2, c_4\}$ basis for Col. Sp. of A

Ex. Find a subset of vectors

$\bar{v}_1 = (0,1,2,0)$, $\bar{v}_2 = (2,1,2,2)$, $\bar{v}_3 = (0,1,2,0)$

$\bar{v}_4 = (-1,0,1,-1)$

that form a basis of their span

Method: Construct matrix A with $\bar{v}_1, \dots, \bar{v}_4$ as columns:

$$A = \begin{array}{c} \begin{pmatrix} 0 & 2 & 0 & -1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 2 & 1 \\ 0 & 2 & 0 & -1 \end{pmatrix} \\ \begin{matrix} \bar{v}_1 & \bar{v}_2 & & \bar{v}_4 \end{matrix} \end{array}$$

Now — find a basis for the Col. Sp.!

Rank and Nullity (§5.6)

Thm. Row and Column Sp. of a matrix A have the same dimension, called the rank of A

Nullity of A = dimension of the Nullspace

Rank + Nullity = number of columns

Ex. Find the rank and the nullity of the matrix

$$A = \begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 1 & 1 & 2 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \tilde{A}$$

Basis for the row space: $\bar{v}_1 = (1, 0, 0, -1, 0)$, $\bar{v}_2 = (0, 1, 2, 1, 0)$

rank(A) = dim. of the row sp. = number of vectors in a basis $= 2$

nullity(A) = number of columns − rank(A)
$= 5 - 2 = 3$

$\Rightarrow$ The solution space is 3-dimensional.

$$\tilde{A}x = 0 \qquad \begin{cases} x_1 - x_4 = 0 \\ x_2 + 2x_3 + x_4 = 0 \end{cases} \qquad \begin{cases} x_1 = s \\ x_2 = -2t - s \\ x_3 = t \\ x_4 = s \\ x_5 = r \end{cases}$$

$3 =$ nullity(A) = nullity$(\tilde{A})$ = number of free parameters t, s, r

$2 =$ rank(A) = rank$(\tilde{A})$ = number of the dependent variables x_1, x_2 $= 2$

If A is $m \times n$ matrix,

then $\text{rank}(A) \leq \min(m, n)$

Overdetermined system: $m > n$, i.e.,

~~the number of equations > number of~~
more equations than unknowns

$$\begin{cases} x_1 - x_2 = 0 \\ x_1 + x_2 = 1 \\ x_1 + 2x_2 = -1 \end{cases} \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{array}\right) = (A|\bar{b})$$

augmented matrix

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 \\ 0 & 2 \\ 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{rank} = 2$$

$\to A\bar{x} = \bar{b}$ cannot be solvable or consistent for all $\bar{b}$ — rank $\leq$ unknown

Underdetermined systems: $m < n$, i.e.
less equations than unknowns

$$= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{rank} \leq \text{number of equations}$$

Consistency

$A\bar{x} = \bar{b}$ consistent $\Leftrightarrow$ $\bar{b}$ is in the column sp. of A

$\Leftrightarrow \text{rank}(A) = \text{rank}(A|\bar{b})$
augmented matrix

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & 3 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & -5/2 \end{array}\right)$$

$\to$ not consistent

Thm. If $\operatorname{rank}(A)$ = number of rows
$\Rightarrow A\bar{x} = \bar{b}$ is consistent for every $\bar{b}$

Only possible if number of rows $\leq$ numb. of co

or ~~less~~ no more equations than unknowns

Otherwise $\longrightarrow$ overdetermined

Inner Products §6.1.

Euclidean Inner Product on R^n = dot product:

~~$\bar{u}\cdot\bar{v} = \langle u, v\rangle$~~ $u_E = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$

$\bar{u}\cdot\bar{v} = \langle\bar{u}, \bar{v}\rangle = u_1v_1 + \dots + u_nv_n$

More generally:
(Weighted) Euclidean Inner Product:

$\langle\bar{u}, \bar{v}\rangle = w_1 u_1 v_1 + \dots + w_n u_n v_n$

$w_1, \dots, w_n$ ~~are~~ weights – positive real numbers

Ex. $\langle\bar{u}, \bar{v}\rangle = 4u_1v_1 + 9u_2v_2$

~~E.g. After coordinate change~~:

~~Start with~~

Even more generally:

An inner product $\langle\bar{u}, \bar{v}\rangle$ for $\bar{u}, \bar{v} \in R^n$ is any scalar function of $\bar{u}, \bar{v}$ satisfying:

(1) $\langle\bar{u}, \bar{v}\rangle = \langle\bar{v}, \bar{u}\rangle$ symmetry

(2) $\langle\bar{u}+\bar{v}, \bar{w}\rangle = \langle\bar{u}, \bar{w}\rangle + \langle\bar{v}, \bar{w}\rangle$ additivity

(3) $\langle k\bar{u}, \bar{v}\rangle = k\langle\bar{u}, \bar{v}\rangle$ homogeneity

(4) $\langle\bar{v}, \bar{v}\rangle \geq 0$ ~~for $\bar{v} \neq 0$~~
and $\langle\bar{v}, \bar{v}\rangle = 0 \iff \bar{v} = 0$

$\langle\bar{u}, \bar{v}\rangle = 3u_1v_1 - u_2v_2$
not an inner produ

Length and Distance wrt Inner Products

Norm or length of a vector $\bar{u}$:

$\|\bar{u}\| = \sqrt{\langle \bar{u}, \bar{u} \rangle}$ (e.g. $\sqrt{u_1^2 + u_2^2}$ in the Euclidean case in $\mathbb{R}^2$

Distance between two points $\bar{u}, \bar{v}$:

$d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\| = \sqrt{\langle \bar{u} - \bar{v}, \bar{u} - \bar{v} \rangle}$

(e.g. $\sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}$ in the Eucl. case

Ex. Find the length of $\bar{u} = (1, -1)$

wrt $\langle \bar{u}, \bar{v} \rangle = u_1 v_1 + 2 u_2 v_2$

$\|\bar{u}\| = \sqrt{\langle \bar{u}, \bar{u} \rangle} = \sqrt{u_1^2 + 2u_2^2} = \sqrt{1 + 2} = \sqrt{3}$

Find the distance betw. $\bar{u} = (1, 1)$ and $\bar{v} = (1, -1)$

$d(\bar{u}, \bar{v}) = \sqrt{\langle \bar{u} - \bar{v}, \bar{u} - \bar{v} \rangle} = \sqrt{\langle 0, 2 \rangle} = \sqrt{0^2 + 2 \cdot 2^2} =$ $\sqrt{8}$

Circle wrt weighted inn. prod. $\rightarrow$ Ellipse

~~Inner Prod. gener. by matrices~~ an inner product

Angle between $\bar{u}, \bar{v}$ wrt $\langle \bar{u}, \bar{v} \rangle$ in a number $0 \le \theta \le \pi$

s.t. $\cos\theta = \dfrac{\langle u, v \rangle}{\|u\| \|v\|}$

Ex. $\bar{u} = (1, 0, -1, 2)$, $\bar{v} = (-1, 3, 0, 1)$, ~~$\langle \bar{u}, \bar{v} \rangle = u_1 v_1$~~

$\langle \bar{u}, \bar{v} \rangle = u_1 v_1 + 2 u_2 v_2 + 3 u_3 v_3 + 4 u_4 v_5$

$\langle \bar{u}, \bar{v} \rangle = 1 \cdot (-1) + 2 \cdot 0 \cdot 3 + 3 \cdot (-1) \cdot 0 + 4 \cdot 1 \cdot 2 = 7$

$\cos\theta =$ ~~$\bar{u}$~~

$\|\bar{u}\| = \sqrt{\langle \bar{u}, \bar{u} \rangle} = \sqrt{1^2 + 0^2 + (-1)^2 + 2^2}$

~~Def~~ $\bar{u}, \bar{v}$ are orthogonal wrt $\langle \bar{u}, \bar{v} \rangle$ if $\langle \bar{u}, \bar{v} \rangle = 0$

Ex. $\bar{u}, \bar{v} \in \mathbb{R}^2$, ~~$\langle \bar{u}, \bar{v} \rangle = (u_1+u_2)(v_1+v_2) +$~~

$\langle \bar{u}, \bar{v} \rangle = (u_1 - u_2)\cdot(v_1 - v_2) + u_1 \cdot v_1$

$\langle \bar{v}, \bar{v} \rangle = (v_1 - v_2)^2 + v_1^2 \geqslant 0 \longrightarrow A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$v_1^2 + (v_2 - v_1)^2$

If $\langle \bar{v}, \bar{v} \rangle = 0 \Rightarrow v_1 = v_2,\ v_1 = 0 \Rightarrow \bar{v} = 0$

~~Inner Pro~~

Here $\langle \bar{u}, \bar{v} \rangle = A\bar{u} \cdot A\bar{v}$,

where $A\bar{u} = \begin{pmatrix} u_1 - u_2 \\ u_1 \end{pmatrix}$, $A\bar{v} = \begin{pmatrix} v_1 - v_2 \\ v_1 \end{pmatrix}$,

So $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \Rightarrow A\bar{u} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 - u_2 \\ u_1 \end{pmatrix}$

General case: $\bar{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$, $\bar{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$

A invertible $n \times n$ matrix

Then: $\langle \bar{u}, \bar{v} \rangle := A\bar{u} \cdot A\bar{v}$ is an inner product

Called " the inn. prod. generated by the matrix,

Ex. $A = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ $\bar{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, $\bar{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$\langle \bar{u}, \bar{v} \rangle = A\bar{u} \cdot A\bar{v} = \begin{pmatrix} u_1 - u_2 \\ u_1 \end{pmatrix} \cdot \begin{pmatrix} v_1 - v_2 \\ v_1 \end{pmatrix}$

$A\bar{u} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1 - u_2 \\ u_1 \end{pmatrix}$, $A\bar{v} = \begin{pmatrix} v_1 - v_2 \\ v_1 \end{pmatrix}$,

Ex. Find the inn prod. gener. by $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$

Orthogonal & Orthonormal Bases wrt Inner Products

~~Given~~ A basis $\{\bar{v}_1, \ldots, \bar{v}_n\}$ is orthonormal (orthogonal) wrt an inner product $\langle \bar{u}, \bar{v} \rangle$ if

$\langle \bar{v}_i, \bar{v}_i \rangle = 1$ for all i and

$\langle \bar{v}_i, \bar{v}_j \rangle = 0$ for all $i \neq j$

Ex. ① $\bar{v}_1 = (1,0)$, $\bar{v}_2 = (0,1)$ standard basis,

$\langle \bar{u}, \bar{v} \rangle = \bar{u} \cdot \bar{v}$ dot product

$\Rightarrow \langle \bar{v}_1, \bar{v}_1 \rangle = 1^2 + 0^2 = 1$, $\langle \bar{v}_2, \bar{v}_2 \rangle \ldots$

② Change the basis: $\bar{v}_1 = (1,0)$, $\bar{v}_2 = \frac{1}{\sqrt{2}}(1,1)$

③ $(0,1,0)$, $(1,0,1)$, $(1,0,-1)$ orthog., not orthon.

③ ~~$v_1 = (1,0)$, $v_2 =$~~

~~$\bar{v}_1 = (0,1,0)$, $\bar{v}_2 = (1,0,1)$, $\bar{v}_3 = (1,0,-1)$~~

$\bar{v}_1 = (0,1,0)$, $\bar{v}_2 = \left(-\frac{4}{5}, 0, \frac{3}{5}\right)$, $\bar{v}_3 = \left(\frac{3}{5}, 0, \frac{4}{5}\right)$

$\frac{1}{\sqrt{5}}$ $\frac{2}{\sqrt{5}}$ $\frac{1}{\sqrt{5}}$ $\frac{2}{\sqrt{5}}$

Coordinate relative to Orthon. Bases:

- If $\{\bar{v}_1, \ldots, \bar{v}_n\}$ is orthon. basis and $\bar{v} = c_1\bar{v}_1 + \ldots + c_n v_n$, then $c_i = \langle \bar{v}, \bar{v}_i \rangle$

— " — orthogonal $\Rightarrow c_i = \dfrac{\langle \bar{v}, \bar{v}_i \rangle}{\|v_i\|^2}$

Ex. 3 $\bar{v} = (1,1,1)$

$c_1 = \langle \bar{v}, \bar{v}_1 \rangle = \langle (1,1,1), (0,1,0) \rangle = 1$

$c_2 = \langle \bar{v}, \bar{v}_2 \rangle = \langle (1,1,1), (-\frac{4}{5}, 0, \frac{3}{5}) \rangle = -\frac{1}{5}$

$c_3 = \langle \bar{v}, \bar{v}_3 \rangle = \ldots 1 \quad \frac{2}{\sqrt{5}} \quad \frac{1}{\sqrt{5}}$

$\bar{v} = 1 \cdot \bar{v}_1 + (-\frac{1}{5}) \bar{v}_2 + 1 \cdot \bar{v}_3$

~~Making~~

Gram- Schmidt Process

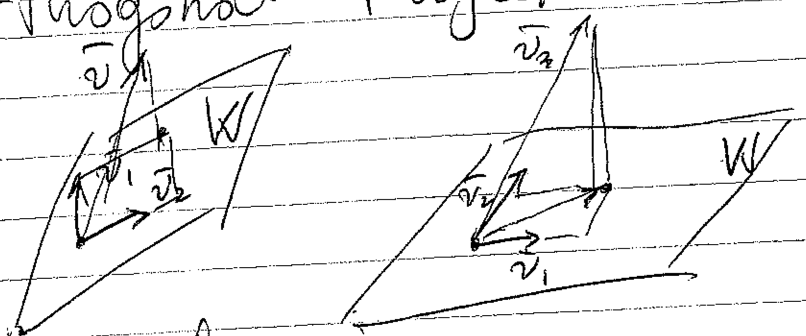
Begin ~~th~~ with general basis $\rightarrow$ Construct an ~~orthonormal~~ orthogonal basis

$\{\bar{u}_1, \ldots, \bar{u}_n\}$ general basis

Construct orthogonal basis of $\{\bar{v}_1, \ldots, \bar{v}_n\}$:

$\bar{v}_1 = \bar{u}_1 \quad , \quad \bar{v}_2 = \bar{u}_2 - \frac{\langle u_2, v_1 \rangle}{\|v_1\|^2} \bar{v}_1$

Orthogonal projection to a subspace W (30)



~~proj~~$_V$

orthogonal projection of $\bar v$ to W is $\mathrm{proj}_V \bar v = c_1 v_1 + c_2 \bar v_2$

- If $\{\bar v_1, \bar v_2\}$ is ~~orthogonal~~, orthonormal,

  $c_1 = \langle \bar v, v_1 \rangle$, $c_2 = \langle \bar v, \bar v_2 \rangle$

- If $\{\bar v_1, v_2\}$ is orthogonal, ~~normalize~~:

  $\cancel{\bar w_1 = \frac{\bar v_1}{\|\bar v_1\|}, \ \bar w_2 = v_2}$

  $c_1 = \frac{\langle v, \bar v_1 \rangle}{\|v_1\|^2}$, $c_2 = \frac{\langle v, \bar v_2 \rangle}{\|\bar v_2\|^2}$

Projection formula $\mathrm{proj}_W \bar v = \sum_{i=1}^{n} \frac{\langle v, \bar v_i \rangle}{\|v_i\|^2} \bar v$

Ex. Find orthog. proj. of $\bar v = (1, 1, 1)$

to ~~the~~ $V = \mathrm{span}\{\bar v_1, \bar v_2\}$,

$\bar v_1 = (0, 1, 0)$, $\bar v_2 = (-1, 0, 2)$

- $\{\bar v_1, \bar v_2\}$ is orthogonal $\Rightarrow \mathrm{proj}_V \bar v = c_1 \bar v_1 + c_2 \bar v_2$

  $c_1 = \frac{\langle \bar v, \bar v_1 \rangle}{\|v_1\|^2} = \frac{1}{1} = 1$, $c_2 = \frac{1}{\sqrt{5}}$

$\Rightarrow \mathrm{proj} = \bar v_1 + \frac{1}{\sqrt{5}} \bar v_2$

Gram - Schmidt Process (§6.3)

$\{\bar{u}_1, \dots, \bar{u}_n\}$ general basis in V

Look for an orthogonal basis $\{\bar{v}_1, \dots, \bar{v}_n\}$

Step 1. $\bar{v}_1 = u_1$

Step 2 $\bar{v}_2 = \bar{u}_2 - \mathrm{proj}_{W_1} \bar{u}_2$, $W_1 = \mathrm{span}\{v_1\}$

So $\bar{v}_2 = \cancel{\bar{u}_2 - c_1 v_1} = \bar{u}_2 - \dfrac{\langle \bar{u}_2, \bar{v}_1 \rangle}{\|v_1\|^2} \bar{v}_1$

Step 3 $\bar{v}_3 = \bar{u}_3 - \mathrm{proj}_{W_2} \bar{u}_3$, $W_2 = \mathrm{span}\{v_1, v_2\}$

So $\bar{v}_3 = \cancel{u_3 - c_1 \bar{v}_1 - c_2 \bar{v}_2} = \bar{u}_3 - \dfrac{\langle u_3, v_1 \rangle}{\|v_1\|^2} \bar{v}_1 - \dfrac{\langle u_3, v_2 \rangle}{\|v_2\|^2}$

. . . .

Step n $\bar{v}_n = \bar{u}_n - \mathrm{proj}_{W_{n-1}} \bar{u}_n$, $W_{n-1} = \mathrm{span}\{v_1, \dots, v_{n-1}\}$

$\cancel{\bar{v}_n = \bar{u}_n - c_1 v_1 - \dots - c_n v_n}$

$$\bar{v}_n = \bar{u}_n - \frac{\langle u_n, \bar{v}_1 \rangle}{\|\bar{v}_1\|^2} \bar{v}_1 - \dots - \frac{\langle u_n, v_{n-1} \rangle}{\|v_{n-1}\|^2} \bar{v}_{n-1}$$

$\to \{\bar{v}_1, \dots, \bar{v}_n\}$ orthogonal basis

Ex. $\bar{u}_1 = (1,1,1)$, $\bar{u}_2 = (0,1,1)$, $u_3 = (0,0,1)$

Step 1 $\bar{v}_1 = u_1 = \boxed{(1,1,1)}$

Step. 2 $\bar{v}_2 = \bar{u}_2 - \dfrac{\langle \bar{u}_2, v_1 \rangle}{\|v_1\|^2} v_1 = (0,1,1) - \dfrac{2}{3}(1,1,1) = \left(-\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right)$

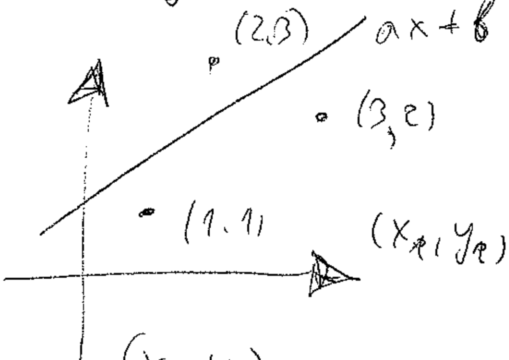
Step 3. $\bar{v}_3 = \bar{u}_3 - \dfrac{\langle \bar{u}_3, \bar{v}_1 \rangle}{\|v_1\|^2} \bar{v}_1 - \dfrac{\langle \bar{u}_3, \bar{v}_2 \rangle}{\|v_2\|^2} \bar{v}_2 = (0,0,1) - \dfrac{1}{3}(1,1,1) - \dfrac{1/3}{2/3}$

$\times \left(-\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{1}{3}\right) = (0, -1/2, 1/2)$

$ax + b = y$

Least Squares

(31.5)

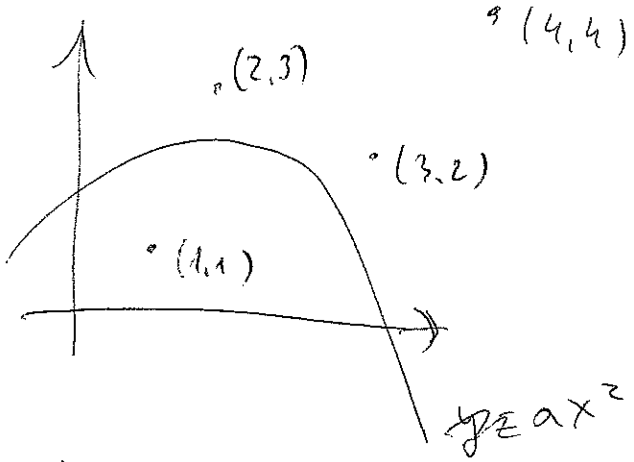


$\cancel{a \neq b} =$

ax

(x, y)	$ax + b = y$
$(1,1)$	$a + b = 1$
$(2,3)$	$2a + b = 3$
$(3,2)$	$3a + b = 2$

$$\begin{cases} a + b = 1 \\ 2a + b = 3 \\ 3a + b = 2 \end{cases}$$



$\cancel{y=} ax^2 + bx + c = y$

(x, y)	$ax^2 + bx + c = y$
$(1,1)$	$a + b + c = 1$
$(2,3)$	$a \cdot 2^2 + b \cdot 2 + c = 3$
$(3,2)$	$a \cdot 3^2 + b \cdot 3 + c = 2$
$(4,4)$	$a \cdot 4^2 + b \cdot 4 + c = 4$

$$\begin{cases} a + b + c = 1 \\ 4a + 2b + c = 3 \\ 9a + 3b + c = 2 \\ 16a + 4b + c = 4 \end{cases}$$

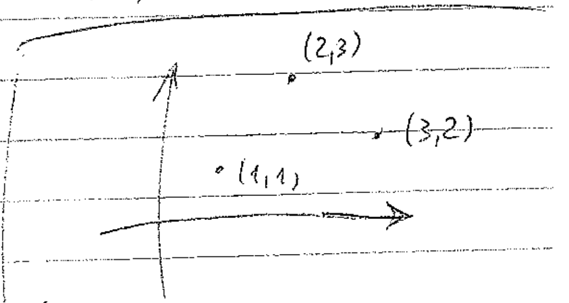
$\longrightarrow$ inconsistent system

Orthonormal basis:

$\{\bar{q}_1, \bar{q}_2, \bar{q}_3\}$ with $\bar{q}_1 = \frac{\bar{v}_1}{\|\bar{v}_1\|} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$\bar{q}_2 = \frac{\bar{v}_2}{\|\bar{v}_2\|} = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

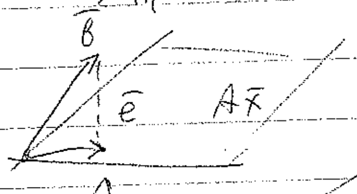
$\bar{q}_3 = \frac{\bar{v}_3}{\|\bar{v}_3\|} = \left(0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$



Best Approximation by the method of Least Squares (Ch. 6.4)

Consider inconsistent systems $A\bar{x} = \bar{b}$, e.g. $\begin{cases} x+y = 1 \\ 2x+2y = -1 \end{cases}$

Idea – minimize the error $\bar{e} = A\bar{x} - \bar{b}$, equivalently its norm squared $\|\bar{e}\| = e_1^2 + \ldots + e_m^2$



$A\bar{x} = \begin{pmatrix} a_{11} \\ \vdots \\ a_{m1} \end{pmatrix} x_1 + \ldots + \begin{pmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{pmatrix} x_n$ varies over the colum space W

$A\bar{x} = \text{proj}_W \bar{b}$

approx solution

Method: Consider and solve the associated normal system:

$A^T A \bar{x} = A^T \bar{b}$

Ex. $\begin{cases} x_1 - x_2 = 4 \\ 3x_1 + 2x_2 = 1 \\ -2x_1 + 4x_2 = 3 \end{cases}$ $\quad \underbrace{\begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}}_{\bar{b}}$, $\quad A^T A = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 3 & 2 \\ -2 & 4 \end{pmatrix}$

$= \begin{pmatrix} 14 & -3 \\ -3 & 21 \end{pmatrix}$, $\quad A^T \bar{b} = \begin{pmatrix} 1 & 3 & -2 \\ -1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$

Normal system: $A^T A \bar{x} = A^T \bar{b}$: $\quad \begin{pmatrix} 14 & -3 \\ -3 & 21 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 10 \end{pmatrix}$

$\left(\begin{array}{cc|c} 14 & -3 & 1 \\ -3 & 21 & 10 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -\frac{3}{14} & \frac{1}{14} \\ -3 & 21 & 10 \end{array}\right) \xrightarrow{\times 3} \left(\begin{array}{cc|c} 1 & -\frac{3}{14} & \frac{1}{14} \\ 0 & 21 - \frac{9}{14} & 10 + \frac{3}{14} \end{array}\right)$

$\longrightarrow \left(\begin{array}{cc|c} 1 & -\frac{3}{14} & \frac{1}{14} \\ 0 & \frac{285}{14} & \frac{143}{14} \end{array}\right)$ $\quad \frac{285}{14} x_2 = \frac{143}{14} \Rightarrow x_2 = \frac{143}{285}$

$x_1 - \frac{3}{14} x_2 = \frac{1}{14} \Rightarrow x_1 = \frac{1}{14} + \frac{3}{14} \cdot \frac{143}{285}$

$= \frac{1}{14}\left(1 + \frac{429}{285}\right) = \frac{714}{14 \cdot 285} = \frac{51}{285} = \frac{17}{95} \Rightarrow (x_1, x_2) = \left(\frac{17}{95}, \frac{1}{}\right.$

, the least square solution

Eigenvalues & Eigenvect. of Matrices

Given: A – square matrix $n \times n$

Eigenvector of A: $\bar{x} \in \mathbb{R}^n$, $\bar{x} \neq 0$, $A\bar{x} = \lambda \bar{x}$

for some scalar λ, called an eigenvalue

~~Problem. Find eigenvalues & eigenv. for~~

Take $A = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}$, $\bar{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$$A\bar{x} = \begin{pmatrix} 3 & 0 \\ 8 & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 3\bar{x}$$

So $A\bar{x} = 3\bar{x}$, $\lambda = 3$ eigenvalue, $\bar{x} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ corresp. eigenvec.

How to find e-vect. & e-values?

$\det\begin{pmatrix} \lambda-3 & 0 \\ -8 & \lambda+1 \end{pmatrix} = (\lambda-3)(\lambda+1) =$ $\lambda = -1$ $\begin{pmatrix} -4 & 0 \\ -8 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}$ $\rightsquigarrow$ write Characteristic polynomial

$p(\lambda) = \det(\lambda I - A)$, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ identity matrix

$$= \det\left[\lambda\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{pmatrix}\right] = \det\begin{pmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -4 & 17 & \lambda-8 \end{pmatrix}$$

$$= \lambda\begin{vmatrix} \lambda & -1 \\ 17 & \lambda-8 \end{vmatrix} + 1\cdot\begin{vmatrix} 0 & -1 \\ -4 & \lambda-8 \end{vmatrix} = \lambda\left(\lambda(\lambda-8) + 1\cdot 17\right)$$

$$+1\left(0(\lambda-8) - (-1)(-4)\right) = \lambda^3 - 8\lambda^2 + 17\lambda - 4$$

~~Eigenv.~~ Rule: Eigenvalues are the roots of the charact. polynomial!