Course 2328 Complex Analysis

Sheet 5

Due: Friday, at the end of the lecture

Exercise 1

Let γ be the sum of two line segments connecting -1 with iy and iy with 1, where y is a fixed parameter.

(i) Write an explicit parametrization for γ;

(ii) For every y, evaluate the integrals $\int_\gamma z\,dz$ and $\int_\gamma \bar{z}\,dz$. Which of the integrals is independent of y?

(iii) Use (ii) to show that the conclusion of Cauchy's theorem does not hold for $f(z) = \bar{z}$.

Solution (i) Use the general formula $\gamma(t) = (1-t)a + tb$, $0 \le t \le 1$, for the line segment connecting a and b.

(ii) The first integral should be independent.

(iii) The second integral depends on the path with fixed endpoints, so the conclusion of the Cauchy's theorem does not hold.

Exercise 2

(i) Calculate $\int_\gamma f(z)\,dz$, where

$$f(z) = z + \frac{1}{z} - \frac{4}{z^3}$$

and $\gamma(t) = ce^{it}$, $0 \le t \le 2\pi$.

(ii) Use (i) to show that $f(z)$ does not have an antiderivative in its domain of definition.

(iii) Does $f(z) = \frac{1}{z^n}$ have an antiderivative, where $n \ge 2$ is an integer?

(iv) Give an example of an open set Ω, where the function $f(z) = \frac{1}{(z+1)(z-1)}$ does not have an antiderivative.

Justify your answer.

Solution

(i) Direct calculation shows that the integral is $2\pi i$, arising from the term $1/z$.

(ii) $f(z)$ cannot have antiderivative, otherwise the former integral would be zero.

(iii) It does.

(iv) For example, any disk around 1 with points ± 1 removed. By Residue Theorem, the integral along small circle around 1 is not zero, hence there is no antiderivative.

Exercise 3

Calculate the residues:

(i) $\mathrm{Res}_3 \frac{z^2 - e^{-z}}{z-3}$;

(ii) $\mathrm{Res}_0 \frac{\sin(z^2) - e^{2z}}{z^5 + z^3 - z}$;

(iii) $\mathrm{Res}_{-1} \frac{\cos(2\pi z^{-2}) + z e^z}{\sin(\pi z)}$.

Solution is analogous to the following:

Exercise 4

Calculate the residues:

(i) $\mathrm{Res}_0 \frac{z^2 - e^z}{z^5 - z}$

Solution

$$\mathrm{Res}_0 \frac{z^2 - e^z}{z^5 - z} = \frac{(z^2 - e^z)|_{z=0}}{(z^5 - z)'_{z=0}} = \frac{-1}{-1} = 1$$

(ii) $\mathrm{Res}_1 \frac{\cos(2\pi z) - z}{e^{z-1} - 1}$

Solution

$$\mathrm{Res}_1 \frac{\cos(2\pi z) - z}{e^{z-1} - 1} = \frac{(\cos(2\pi z) - z)|_{z=1}}{(e^{z-1} - 1)'_{z=1}} = \frac{1-1}{1} = 0.$$

Exercise 5

Evaluate the integrals:

(i) $\int_{|z|=2} \frac{e^z}{(z^2+z)(z-3)}\, dz$;

(ii) $\int_0^{2\pi} \frac{1}{(2-\sin\theta)(3+\cos\theta)} d\theta$;

(iii) $\int_{-\infty}^{\infty} \frac{x^3 - x}{x^6 + 1} dx$;

(iv) $\int_{-\infty}^{\infty} \frac{x^3 e^{i\lambda x}}{x^4 + 1} dx$, $\lambda > 0$;

(v) $\int_0^{\infty} \frac{x^\alpha}{x^2 + 1} dx$, $-1 < \alpha < 1$, $\alpha \notin \mathbb{Z}$.

Solution

(i) Using Residue Theorem, we have singuarities at $z = 0, -1, 3$, from which only 0 and -1 are in the disk. The integral is the sum of residues at those points.

(ii) Use the substitution

$$\mathsf{cos}\theta = \frac{1}{2}(z + \frac{1}{z}), \quad \mathsf{sin}\theta = \frac{1}{2i}(z - \frac{1}{z}), \quad d\theta = \frac{dz}{iz}$$

and then the Residue Thm.

(iii) The function under the integral is rational and admits holomorphic extension to

$$f(z) = \frac{z^3 - z}{z^6 + 1}$$

that has singularities at the 6th order roots from -1. Three of these roots, a_1, a_2, a_3 are in the upper half-plane. It remains to show that $zf(z) \to 0$ as $z \to \infty$ and compute the sum of Residues at the a_j's.

(iv) This is the Fourier type integral of $f(x)e^{i\lambda x}$. The function f admits holomorphic extension with singularities at the 4th roots from -1. The Residue theorem is used for the rectangular region, so the asymptotics $f(z) \to 0$, as $z \to \infty$ is to be checked. The result is the sum of the residues in the upper half-plane.

(v) This integral is computed via the annulus $r < |z| < R$ with cut along the positive real axis. The real axis part contributes with coefficient $(1 - e^{2\pi\alpha}$, since the power function z^α has different boundary values from above and from below. The final integral is the sum of the residues divided by that coefficient.